Complete tripartite graphs and their competition numbers

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Abstract

We present a piecewise formula for the competition numbers of the complete tripartite graphs. For positive integers $x$, $y$, and $z$ where $2 \leq x \leq y \leq z$, the competition number of the complete tripartite graph $K_{x,y,z}$ is $yz - z - y - x + 3$ whenever $x \neq y$ and $yz - 2y - z + 4$ otherwise.

1 Introduction

In this note we consider competition graphs as introduced by Cohen in [1] and we consider a problem left open by Kim and Sano in [3]. Let $D$ be a digraph with vertex set $V$ and arc set $A$. If $u, v \in V$ have a common out-neighbor in $D$, then $u$ and $v$ are said to be in competition. The simple graph $(V, E)$ in which edge set $E$ is defined as

$$E = \{\{u, v\} : \text{u and v are in competition in } D\}$$

is called the competition graph of $D$ and is denoted $C(D)$. Given the applicative nature of competition graphs (one example is that $V$ represents a set of organisms in a food-web and competition is defined by organisms competing for food), it is important to ask which graphs are competition graphs of acyclic digraphs. In [8], Roberts observed that for any graph $G$ and for a sufficiently large integer $k$, $G \cup I_k$ is the competition graph of an acyclic digraph, where $I_k$ denotes the graph on $k$ isolated vertices. The minimum such $k$ is called the competition number of $G$. Formally, the competition number of $G$ is

$$k(G) = \min\{k : G \cup I_k = C(D) \text{ in which } D \text{ is an acyclic digraph}\}.$$ 

In general, the problem of computing $k(G)$ is NP-hard [5]. So to reduce generality, $G$ will belong to the class of complete multipartite graphs. The following theorems are what is currently known concerning the competition numbers of complete multipartite graphs.

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Theorem 1.1 The competition number of the complete bipartite graph $K_{n_1,n_2}$ is $n_1n_2 - n_1 - n_2 + 2$.

Theorem 1.1 is a corollary of the statement that if $G$ is a triangle-free connected graph, then $k(G) = |E(G)| - |V(G)| + 2$. Recently, Kim and Sano [3] found the competition number of the complete tripartite graph $K_{n,n,n}$.

**Theorem 1.2** The competition number $k(K^3_n)$ is $n^2 - 3n + 4$.

We extend Kim and Sano’s result to complete tripartite graphs in which the partite sets may not have equal size. We prove the following formula:

**Theorem 1.3** For positive integers $x$, $y$ and $z$ where $2 \leq x \leq y \leq z$,

$$k(K_{x,y,z}) = \begin{cases} 
yz - 2y - z + 4, & \text{if } x = y \\
yz - y - x + 3, & \text{if } x \neq y
\end{cases}$$

Some progress has been made on competition numbers of the complete tetrapartite graph $K^4_n$ [4] and, more generally, the complete multipartite graph $K^m_n$ [4].

**Theorem 1.4** If $n \geq 5$ is odd, then

$$n^2 - 4n + 7 \leq k(K^4_n) \leq n^2 - 4n + 8.$$  

**Theorem 1.5** If $n$ is prime and $m \leq n$, then

$$k(K^m_n) \leq n^2 - 2n + 3.$$  

Park et al. [7] give bounds for the general case with respect to $L(n)$, the largest size of a family of mutually orthogonal latin squares of order $n$.

**Theorem 1.6** If $m$ and $n$ are positive integers such that $3 \leq m \leq L(n) + 2$, then

$$k(K^m_n) \leq n^2 - n + 1.$$  

For small values of $n$, Park et al. [6] found the following competition numbers.

**Theorem 1.7** If $m \geq 2$, then $k(K^m_2) = 2$ and if $m \geq 3$, then $k(K^m_3) = 4$.

While we do not do so in this paper, it would be interesting to study the competition number of $K_{n_1,n_2,n_3,n_4}$ since very little is currently known. Furthermore, there remains much to be known on computing the competition number $k(K^m_n)$.
2 Edge clique covers of $K_{x,y,z}$

Let $U = \{u_1, \ldots, u_x\}$, $V = \{v_1, \ldots, v_y\}$, and $W = \{w_1, \ldots, w_z\}$ be the vertex partition sets of $K_{x,y,z}$ where $2 \leq x \leq y \leq z$. We use $\Delta(i,j,k)$ to denote the clique induced on the vertex set $\{u_i, v_j, w_k\}$ and we use $\Delta(j,k)$ to denote the clique induced on the vertex set $\{v_j, w_k\}$. Note that a clique of order 3 is the largest clique in $K_{x,y,z}$.

Competition numbers can be computed by first finding a minimal edge clique cover. Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a family of cliques in a graph $G$; i.e. the subgraph induced on $S_i \subseteq V(G)$ is complete for each $i \in [m]$. The family $\mathcal{S}$ is called an edge clique cover of $G$ provided $\{u,v\} \in E(G)$ if and only if $\{u,v\} \subseteq S_i$ for some $i \in [m]$. The edge clique cover number of $G$, denoted $\theta_e(G)$, is

$$\theta_e(G) = \min\{|\mathcal{S}| : \mathcal{S} \text{ is an edge clique cover of } G\}.$$ 

Certainly, for any graph $G$, $k(G) \leq \theta_e(G)$. Indeed, if $\theta_e(G) = k$, then each vertex of a clique in $G$ can be directed to a vertex of $I_k$ in the digraph $D$.

We find a minimal edge clique cover of $K_{x,y,z}$ using $r$-semi latin squares. An $r$-semi latin square of order $n$ is an $n \times n$ array such that each element (or symbol) from the set $S = \{s_1, s_2, \ldots, s_n\}$ appears in each row and each column, and each cell contains $r$ elements. If we label the rows and columns with sets $R = \{r_1, r_2, \ldots, r_n\}$ and $C = \{c_1, c_2, \ldots, c_n\}$ respectively, we may think of an $r$-semi latin square as a set of ordered triples $(r_i, c_j, s_k)$, where symbol $s_k$ appears at the intersection of row $r_i$ and column $c_j$. Where convenient, we use the notation $c_j \circ s_k$ to denote the row containing symbol $s_k$ in column $c_j$.

Henceforth $q$ and $r$ are positive integers such that $z = qy + r$, where $0 \leq r < y$. Let $L$ be a $(q+1)$-semi latin square of order $y$ on the symbol set $S = \{s_1, \ldots, s_{(q+1)y}\}$. Furthermore, let $R' = \{r'_1, \ldots, r'_x\} \subseteq R$ be a set of $x$ rows and let $S' = \{s'_1, \ldots, s'_z\} \subseteq S$ be a set of $z$ symbols. We use

$$L(R', C, S') = \{(r'_i, c_j, s'_k) : (r'_i, c_j, s'_k) \in L, r'_i \in R', s'_k \in S'\}$$

to denote the $x \times y$ array on symbol set $S'$ induced by the intersection of rows $R'$ and columns $C$. Note that the family $\mathcal{F}$, defined below, is a subset of an edge clique cover of $K_{x,y,z}$. In fact, we will later show that $\mathcal{F}$ is a minimal edge clique cover of $K_{x,y,z}$.

$$\mathcal{F} = \{\Delta(i,j,k) : (r'_i, c_j, s'_k) \in L(R', C, S')\} \cup$$

$$\{\Delta(j,k) : (c_j \circ s_k, c_j, s_k) \in L(R \setminus R', C, S')\} \quad (1)$$

For an example of (1), consider $K_{2,4,6}$. Since $z = 6$ and $y = 4$, $q = 1$. We use the following 2-semi latin square of order 4 as $L$ and set $R' = \{r_1, r_4\}$ and $S' = \{s_1, \ldots, s_6\}$, where $r'_1 = r_1$, $r'_2 = r_4$ and $s'_i = s_i = i$ for $1 \leq i \leq 6$. 


Then the rectangular array $L(R', C, S')$ is

\[
\begin{array}{cccc}
1,2 & 4,5 & 3,7 & 6,8 \\
5,6 & 7,8 & 1,2 & 3,4 \\
7,8 & 2,3 & 4,6 & 1,5 \\
3,4 & 1,6 & 5,8 & 2,7 \\
\end{array}
\]

The clique $\Delta(1, 1, 2)$ is included in $\mathcal{F}$ since \((r'_{11}, c_1, s'_2) \in L(R', C, S')\). The same can be said of $\Delta(2, 1, 3)$ since \((r'_{21}, c_1, s'_3) \in L(R', C, S')\). Also, since \((r_2, c_1, s'_5) \in L(R \setminus R', C, S')\), $\Delta(1, 5) \in \mathcal{F}$. The remaining members of $\mathcal{F}$ are given in the following family;

\[
\mathcal{F} = \{\Delta(1, 1, 1), \Delta(1, 1, 2), \Delta(1, 2, 4), \Delta(1, 2, 5), \Delta(1, 3, 3), \Delta(1, 4, 6), \\
\Delta(2, 1, 3), \Delta(2, 1, 4), \Delta(2, 2, 1), \Delta(2, 2, 6), \Delta(2, 3, 5), \Delta(2, 4, 2), \\
\Delta(1, 5), \Delta(1, 6), \Delta(3, 1), \Delta(3, 2), \Delta(4, 3), \Delta(4, 4), \Delta(2, 2), \\
\Delta(2, 3), \Delta(3, 4), \Delta(3, 6), \Delta(4, 1), \Delta(4, 5)\}
\]

**Lemma 2.1** The family $\mathcal{F}$ is an edge clique cover of $K_{x,y,z}$. Moreover, $\mathcal{F}$ is minimal and $\theta_e(K_{x,y,z}) = yz$.

**Proof:** First, we show that $\mathcal{F}$ is an edge clique cover of $K_{x,y,z}$. Let $R' = \{r'_{11}, \ldots, r'_{z} \} \subseteq R$ be a set of $x$ rows and let $S' = \{s'_1, \ldots, s'_z\}$ be a set of $z$ symbols in a $(q+1)$-semi latin square $L$ of order $y$. Consider the edge $e = \{u_i, v_j\}$ in $K_{x,y,z}$, $i \in [x]$ and $j \in [y]$. Let $S_{i,j}$ denote the set of $q + 1$ symbols at the intersection of $r'_i$ and $c_j$. If $S_{i,j} \cap S' = \emptyset$, then $q + 1 \leq q - r$, a contradiction as $r \geq 0$. Therefore there is an integer $k$ such that \((r'_i, c_j, s'_k) \in L(R', C, S')\). Thus the clique $\Delta(i, j, k) \in \mathcal{F}$ covers the edge $e$.

Now set $e = \{u_i, w_j\}$, $i \in [x]$ and $j \in [z]$. Since each symbol of $S'$ appears in each row of $L(R', C, S')$, there is an integer $k$ such that \((r'_i, c_k, s'_j) \in L(R', C, S')\). Hence $\Delta(i, k, j) \in \mathcal{F}$ covers $e$. Finally, set $e = \{v_i, w_j\}$, $i \in [y]$ and $j \in [z]$. There is an integer $k \in [y]$ so that $r_k = c_i \circ s'_j$. If $r_k \in R'$, then certainly $e$ is covered by a clique of order three in $\mathcal{F}$. Otherwise $r_k \in R \setminus R'$ and $\Delta(i, j)$ covers $e$.

We finish the proof by showing that $yz$ is a lower and upper bound for $\theta_e(K_{x,y,z})$. Note that there are $yz$ edges of the form $\{v, w\}$ where $v \in V$ and $w \in W$. Furthermore, there is no clique in $K_{x,y,z}$ that contains two edges of the form $\{v, w\}$. It follows that at least $yz$ cliques are needed to cover the edges that contain end vertices in partitions $V$ and $W$. Hence $\theta_e(K_{x,y,z}) \geq yz$. To show that $yz$ is an upper bound for $\theta_e(K_{x,y,z})$, we need only to provide an edge clique cover of $K_{x,y,z}$ whose cardinality is $yz$. From above, $\mathcal{F}$ is an edge clique cover of $K_{x,y,z}$. Since $L$ contains precisely $y^2(q + 1)$ triples and since symbols from $S \setminus S'$ appear precisely $y$ times in $L$, $\mathcal{F}$ is made of

\[
y^2(q + 1) - y(y(q + 1) - z) = yz
\]
triples. Hence \( \theta_e(K_{x,y,z}) \leq yz \). Moreover, this shows that \( \mathcal{F} \) is a minimal edge clique cover of \( K_{x,y,z} \).

To end this section we comment on a general minimal edge clique cover of \( K_{x,y,z} \) when \( x = y \).

**Lemma 2.2** Let \( S \) be a minimal edge clique cover of \( K_{y,y,z} \) and let \( S, S' \in S \). If \( |S \cap S'| = 2 \), then \( S = \{u, v, w\} \) and \( S' = \{u, v, w'\} \) where \( u \in U \), \( v \in V \) and \( w, w' \in W \).

**Proof:** We may assume that \( |S| = 3 \) for each \( S \in S \). Let \( S, S' \in S \) such that \( |S \cap S'| = 2 \).

Since \( |S| = yz \) (by Lemma 2.1) and since no clique contains two edges of the form \( \{v, w\} \), no two cliques of \( S \) share an edge of the form \( \{v, w\} \) where \( v \in V \) and \( w \in W \). Similarly, no two cliques share an edge of the form \( \{u, w\} \) where \( u \in U \) and \( w \in W \). Hence \( (S \cup S') \setminus (S \cap S') \subseteq W \). \( \square \)

## 3 Proof of main result

The following characterization of competition graphs [2] is used to show a lower bound for \( k(K_{x,y,z}) \).

**Theorem 3.1** A graph \( G \) is the competition graph of an acyclic digraph if and only if there exists an ordering \( a_1, \ldots, a_n \) of the vertices of \( G \) and an edge clique cover \( \{S_1, \ldots, S_n\} \) of \( G \) such that if \( a_i \in S_j \), then \( i < j \).

An equivalent way of stating Theorem 3.1 is to say that there exists an ordering \( a_1, \ldots, a_n \) of the vertices of \( G \) and an edge clique cover \( \{S_1, \ldots, S_n\} \) of \( G \) such that \( S_i \subseteq \{a_1, \ldots, a_{i-1}\} \) for each \( i \).

**Theorem 3.2** For integers \( x, y \) and \( z \) where \( 2 \leq x \leq y \leq z \),

\[
k(K_{x,y,z}) \geq yz - z - y - x + 3.
\]

Moreover, if \( x = y \), then

\[
k(K_{y,y,z}) \geq yz - 2y - z + 4.
\]

**Proof:** Let \( k = k(K_{x,y,z}) \) and let \( D \) denote an acyclic digraph such that \( C(D) = K_{x,y,z} \cup I_k \). Note that \( S \) is an edge clique cover of \( K_{x,y,z} \) if and only if \( S \) is an edge clique cover of \( K_{x,y,z} \cup I_k \). Then, from Theorem 3.1, there is an ordering \( a_1, \ldots, a_{x+y+z+k} \) of the vertices of \( K_{x,y,z} \cup I_k \) and an edge clique cover \( S = \{S_1, \ldots, S_{x+y+z+k}\} \) of \( K_{x,y,z} \) such that \( S_i \subseteq \{a_1, \ldots, a_{i-1}\} \) for each \( i \). We may assume that the order of each non empty clique in \( S \) is three. Then \( S_1 = S_2 = S_3 = \emptyset \) and so, by Lemma 2.1, \( |S \setminus \{S_1, S_2, S_3\}| \geq yz \). Hence \( x + y + z + k - 3 \geq yz \) and so \( k \geq yz - x - y - z + 3 \).
Suppose now that \( x = y \) and that, for the sake of contradiction, \( k = yz - 2y - z + 3 \). Then \( S_i \) is non empty for each \( i \geq 4 \), \( S_4 = \{a_1, a_2, a_3\} \) and \( S_5 \subseteq \{a_1, a_2, a_3, a_4\} \). So it must be that \( |S_4 \cap S_5| = 2 \). Without loss of generality, assume that \( S_5 = \{a_2, a_3, a_4\} \). By Lemma 2.2, \( a_1, a_4 \in W \). Let \( l \geq 4 \) be the largest integer such that \( S_{l+1} = \{a_2, a_3, a_4\} \) and \( a_l \in W \). Then \( S_{l+2} = \{a_2, a_j, a_{l+1}\} \) or \( S_{l+2} = \{a_3, a_j, a_{l+1}\} \), \( j \in [l] \setminus \{2, 3\} \). In either case \( |S_{l+2} \cap S_1| = 2 \) or \( |S_{l+2} \cap S_{l+1}| = 2 \). But \( a_{l+1} \in U \cup V \), contradicting Lemma 2.2. Hence \( k \geq yz - 2y - z + 4 \). \( \square \)

We now proceed to the main result. Henceforth \( L \) is a \((q+1)\)-semi latin square of order \( y \) such that \((r_i, c_j, s_k) \in L\) if and only if \( i + j - 1 \equiv k \mod y \). Furthermore, we set \( R' = \{r_1, \ldots, r_{x-1}, r_y\} \) and \( S' = \{s_1, \ldots, s_z\} \) where \( r'_i = r_i \) for \( i \in [x-1] \), \( r'_x = r_y \) and \( s'_i = s_i \) for \( i \in [z] \). For \( y = 5 \) and \( z = 13 \), the arrays below are \( L \) and \( L(R', C, S') \) respectively.

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<thead>
<tr>
<th>1,6,11</th>
<th>2,7,12</th>
<th>3,8,13</th>
<th>4,9,14</th>
<th>5,10,15</th>
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**Proof of Theorem 1.3.** Case 1: \( x = y \).

In this case \( r'_i = r_i \) for each \( i \). We first order the vertices \( a_1, \ldots, a_{2y+z} \) of \( K_{y,y,z} \) as

\[
\begin{align*}
&u_1, v_1, w_1, u_2, v_y, w_y, u_y, v_2, w_2, u_{y-1}, v_{y-1}, w_{y-1}, \ldots, u_3, v_3, w_3, w_{y+1}, \ldots, w_z.
\end{align*}
\]

Note that all vertices of \( K_{y,y,z} \) appear in the vertex ordering. Next, we order \( 2y + z - 3 \) cliques of \( F \) in the following way. The first 6 cliques are ordered as

\[
\begin{align*}
\Delta_1 &= \{u_1, v_1, w_1\}, \\
\Delta_2 &= \{u_2, v_y, w_1\}, \\
\Delta_3 &= \{u_1, v_y, w_y\}, \\
\Delta_4 &= \{u_y, v_1, w_y\}, \\
\Delta_5 &= \{u_y, v_2, w_1\}, \\
\Delta_6 &= \{u_1, v_2, w_2\}
\end{align*}
\]

For \( 0 \leq s \leq y - 4 \), the next \( 3y - 9 \) cliques are given as

\[
\begin{align*}
\Delta_{3s+7} &= \{u_{y-s-1}, v_2, w_{y-s}\}, \\
\Delta_{3s+8} &= \{u_2, v_{y-s-1}, w_{y-s}\}, \text{ and} \\
\Delta_{3s+9} &= \{u_1, v_{y-s-1}, w_{y-s-1}\}.
\end{align*}
\]
Finally, for $0 \leq s \leq z - y - 1$, the remaining $z - y$ cliques are
\[ \Delta_{3y-2+s} = \{u, v, w_{y+s+1}\}, \]
where $u \in U$ and $v \in V$ are any vertices such that $\{u, v, w_{y+s+1}\} \in \mathcal{F}$.

Given the vertex and clique orderings above, we construct a digraph that shows that $yz - 2y - z + 4$ is an upper bound for $k(K_{y,y,z})$. We must first note that
\[ \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_i \subseteq \{a_1, a_2, \ldots, a_{i+3}\} \tag{2} \]
for $i \in [2y + z - 3]$. This follows from the fact that $\Delta_1 = \{a_1, a_2, a_3\}$, $\Delta_2 \setminus \Delta_1 = \{a_4, a_5\}$ and $\Delta_i \setminus (\Delta_1 \cup \ldots \cup \Delta_{i-1}) = \{a_i, a_{i+3}\}$, where $3 \leq i \leq 2y + z - 3$.

Since $\mathcal{F}$ is a minimal edge clique cover, there are $yz - 2y - z + 3$ cliques in $\mathcal{F} \setminus \{\Delta_1, \ldots, \Delta_{2y+z-3}\}$. Set $\mathcal{F} \setminus \{\Delta_1, \ldots, \Delta_{2y+z-3}\} = \{T_1, \ldots, T_{yz-2y-z+3}\}$. Let $D$ be the digraph with the following vertex set $V$ and arc set $A$:
\[
V(D) = \{a_1, \ldots, a_{z+2y-3}\} \cup \{\alpha_0, \ldots, \alpha_{yz-2y-z+3}\}
\]
\[
A(D) = \bigcup_{i=1}^{2y+z-4} \{(\delta, a_{i+4}) : \delta \in \Delta_i\} \cup \{(\delta, \alpha_0) : \delta \in \Delta_{2y+z-3}\} \cup
\bigcup_{i=1}^{yz-2y-z+3} \{(\delta, \alpha_i) : \delta \in T_i\}.
\]

From statement (2), the digraph $D$ is acyclic. Because every clique in $\mathcal{F}$ has a common out-neighbor in $D$, $E(C(D)) \subseteq E(K_{y,y,z})$. Moreover, the in-neighborhood of a vertex of $D$ is a clique in $\mathcal{F}$. Therefore $E(K_{y,y,z}) \subseteq E(K_{y,y,z})$. It follows that $C(D) = K_{y,y,z} \cup I_{yz-2y-z+4}$. Hence, by Theorem 3.2,
\[ k(K_{y,y,z}) = yz - 2y - z + 4. \]

Case 2: $x < y$.

In this case $r'_i = r_i$ for each $i \in [x-1]$ and $r'_x = r_y$. As in the previous case, we begin by ordering vertices and cliques in $K_{x,y,z}$. The vertex ordering $a_1, \ldots, a_{x+y+z}$ of $K_{x,y,z}$ is
\[
u, v_1, w_y, v_2, w, u_1, w_2, u_2, v_y, w_{y-1}, v_{y-1}, \ldots, w_x, v_x,
\]
\[
u, v_{x-1}, w_{x-1}, \ldots, u_3, v_3, w_3, w_{y+1}, \ldots, w_z.
\]
Next, we order $x + y + z - 2$ cliques of $\mathcal{F}$. The first 7 cliques are ordered as
\[
\Delta_1 = \{u_x, v_1, w_y\}, \quad \Delta_2 = \{v_2, w_y\}, \quad \Delta_3 = \{u_x, v_2, u_1\},
\]
\[
\Delta_4 = \{u_1, v_1, w_1\}, \quad \Delta_5 = \{u_1, v_2, u_2\}, \quad \Delta_6 = \{u_2, v_1, w_2\}, \quad \Delta_7 = \{u_2, v_y, w_1\}
\]
For $0 \leq s \leq y - x - 1$, the next $2(y - x)$ cliques in the ordering are given as
\[
\Delta_{2s+8} = \{u_x, v_{y-s}, w_{y-s-1}\}, \quad \text{and} \quad \Delta_{2s+9} = \{u_1, v_{y-s-1}, w_{y-s-1}\}.
\]
For $0 \leq s \leq x - 4$, the next $3x - 9$ cliques are given as

\[
\Delta_{3s+2(y-x)+8} = \{u_{x-s-1}, v_2, w_{x-s}\}, \\
\Delta_{3s+2(y-x)+9} = \{u_2, v_{x-s-1}, w_{x-s}\}, \text{ and} \\
\Delta_{3s+2(y-x)+10} = \{u_1, v_{x-s-1}, w_{x-s-1}\}.
\]

Finally, for $0 \leq s \leq z - y - 1$, the remaining $z - y$ cliques in the ordering are

\[
\Delta_{2y+x-1+s} = \{u, v, w_{y+s+1}\},
\]

where $u \in U$ and $v \in V$ are any vertices such that \(\{u, v, w_{y+s+1}\} \in \mathcal{F}\).

In this case note that

\[
\Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_i \subseteq \{a_1, a_2, \ldots, a_{i+2}\}
\]

for $i \in [z + y + x - 4]$. Set \(\mathcal{F}' \setminus \{\Delta_1, \ldots, \Delta_{z+y+x-4}\} = \{T_1, \ldots, T_{yz-z-y-x+4}\}\). Let $D$ be the digraph with the following vertex set $V$ and arc set $A$:

\[
V(D) = \{a_1, \ldots, a_{z+y+x-4}\} \cup \{a_0, \ldots, a_{yz-z-y-x+2}\}
\]

\[
A(D) = \bigcup_{i=1}^{z+y+x-3} \{(\delta, a_{i+3}) : \delta \in \Delta_i\} \cup \{(\delta, a_0) : \delta \in \Delta_{z+y+x-4}\} \cup \bigcup_{i=1}^{yz-z-y-x+2} \{(\delta, a_i) : \delta \in T_i\}.
\]

It follows from statement (3) that $D$ is acyclic. Furthermore $C(D) = K_{x,y,z} \cup I_{yz-z-y-x+3}$. Hence, by Theorem 3.2,

\[
k(K_{x,y,z}) = yz - z - y - x + 3.
\]

References


