Unavoidable pairs of partial latin squares of order four

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Abstract

This technical report characterizes unavoidable pairs of partial latin squares of order 4 on two symbols.

1 Introduction

A partial latin square of order n is an $n \times n$ array of n distinct symbols in which each symbol occurs at most once in each row and column. If there are no empty cells, then the array is called a latin square. It is useful to think of a partial latin square P as a set of ordered triples, where $(i, j, k) \in P$ if and only if symbol k appears in cell (i, j) of P (see Figure 1).

	1	2	
2		1	
1	2		

Figure 1: $P = \{(2, 2, 1), (2, 3, 2), (3, 1, 2), (3, 3, 1), (4, 1, 1), (4, 2, 2)\}$.

We say that P is avoidable if for each set of n symbols, there exists a latin square L such that if $(i, j, k) \in P$, then $(i, j, k) \notin L$. Unless otherwise stated, we will assume that the symbol set is $\{1, 2, \ldots, n\}$. We say that a pair of partial latin squares $\{P_1, P_2\}$ is avoidable if there is a latin square L that avoids P_1 and P_2 simultaneously. In this paper we characterize the unavoidable pairs of partial latin squares of order 4 on two symbols.

A conjugate of P is an array made by uniformly permuting the coordinates in each of the elements of P. The column/symbol-conjugate of P is an array where the second and third coordinates of each triple are exchanged. An isotope of P is an array formed by relabeling the rows and/or columns and/or symbols of P. The arrays in Figure 2 are examples of an isotope and the column/symbol-conjugate of a given partial latin square.

The following observations are well-known and used throughout this paper.

* P is avoidable if and only if a conjugate of P is avoidable.

* P is avoidable if and only if an isotope of P is avoidable.

			1
	1	2	
2		1	
1	2		

	1	2	1
			1
2		1	
1	2		

4	
2	3
3	1
1	2

Figure 2: Partial latin square P, an isotope of P (row 1 and row 2 are interchanged) and the column/symbol-conjugate of P respectively.

Let P_1 and P_2 be partial latin squares of order 4 on the symbol set $\{1, 2\}$ (i.e. symbols 3 and 4 do not appear in P_1 and P_2). The array formed by superimposing P_1 onto P_2 is called a partial 2-entry latin square of order 4 on the symbol set $\{1, 2\}$. Thus in a partial 2-entry latin square each symbol appears at most twice in a row and column, and each cell contains at most two symbols. Avoiding a partial 2-entry latin square is tantamount to avoiding a pair of partial latin squares. Our chief methodology to show that a partial 2-entry latin square is avoidable (or unavoidable) is to show that its column/symbol-conjugate is avoidable (or unavoidable). Thus it is important to note that the column/symbol-conjugate of a partial 2-entry latin square of order 4 on the symbol set $\{1, 2\}$ is a partial 4×2 2-entry latin square on the symbol set $\{1, 2, 3, 4\}$. Figure 3 contains an example of this.

			12
	12	2	
12		1	
1	2		

4	4
2	23
13	1
1	2

Figure 3: Partial 2-entry latin square and its column/symbol-conjugate respectively.

Results of Chetwynd and Rhodes [4], Cavenagh [2], and Öhman [5], show that every partial latin square of order at least 4 is avoidable. As noted above, we ask for which pairs of partial latin squares of order 4 are avoidable. In this way we continue work begun in [3] on avoiding 2-entry arrays. It is worth noting that for more general 2-entry arrays, Casselgren proved that avoiding such arrays is NP-complete, even in the case when only two distinct symbols occur [1].

2 Main Result

We use $\mathcal{P}_4$ to denote the set of partial 2-entry latin squares of order 4 on the symbol set $\{1, 2\}$ and we use $\mathcal{P}_{4\times 2}$ to denote the set of column/symbol-conjugates of elements in $\mathcal{P}_4$. We present the following propositions without proof, as their proofs are trivial.

Proposition 1 *Let $P \in \mathcal{P}_{4\times 2}$ such that $(i, j, a), (i, j, b), (k, j, a), (k, j, b) \in P$ for some $a, b, i, k \in [4]$ and $j \in [2]$. Then P is avoidable.*

3	2
2	3
13	13
12	12

23	23
13	13
12	12

Figure 4: Arrays Q_1 and Q_2 respectively

Proposition 2 *The arrays Q_1 and Q_2 in Figure 1 are unavoidable.*

We use the notation $\mathcal{Q}$ for the set

$$\{P \in \mathcal{P}_{4\times 2} : P \text{ contains an isotope and/or row/symbol-conjugate of } Q_1 \text{ or } Q_2\}.$$

The main theorem in this section shows that $\mathcal{Q}$ contains all the unavoidable partial 4×2 2-entry latin rectangles on $[4]$. The process by which we show this is described in the following paragraph.

Let $P \in \mathcal{P}_{4\times 2}$. We attempt to construct a 4×2 latin rectangle that avoids P in the following manner. Extract an array $P_1 \in \mathcal{P}_{4\times 2}$ from P such that $P_1 \subseteq P$ and there is a latin rectangle L_1 that avoids P_1. If L_1 avoids P, then we are done. Otherwise determine which entries in P preclude L_1 from avoiding P and thereby construct $P_2 \in \mathcal{P}_{4\times 2}$ such that $P_1 \subseteq P_2 \subseteq P$ where P_2 has exactly one more symbol than P_1, and L_1 does not avoid P_2. Now construct a latin rectangle L_2 that avoids P_2. If L_2 avoids P, then we are done. Otherwise determine which entries in P preclude L_2 from avoiding P and thereby construct $P_3 \in \mathcal{P}_{4\times 2}$ such that $P_1 \subseteq P_2 \subseteq P_3 \subseteq P$ where P_3 contains exactly one more symbol than P_2, and L_2 does not avoid P_3. Now construct a latin rectangle L_3 that avoids P_3. Continue this process until there is an integer m such that either L_m avoids P or $P_m \in \mathcal{Q}$. We express the above process symbolically as

$$P_1 \subseteq P_2 \subseteq \ldots \subseteq P_m$$

$$L_1 \ , L_2 \ , \ldots , L_m$$

If, in the sequence $L_1, L_2, \ldots, L_m$, a partial latin rectangle is given instead of a latin rectangle, we mean for the reader to understand that there are completions of the partial latin rectangle that avoid the corresponding partial 2-entry latin rectangle. In the next two proofs, when we add symbols to P, we add such that P remains a partial 4×2 2-entry latin rectangle. Certainly if P with symbols added is avoidable, then P with no symbols added is avoidable.

Lemma 1 *Let $a, b \in [4]$, $i, j \in [2]$, and $P \in \mathcal{P}_{4\times 2}$ such that*

$$(i, j, a), (i, j + 1, a), (i, j, b), (i, j + 1, b) \in P.$$

If $P \notin \mathcal{Q}$, then P is avoidable.

Proof: Suppose that $P \notin \mathcal{Q}$. Without loss of generality we may assume that $i = 4$ in the statement of the lemma. Because there are 6 possible entries in each column of P outside of row 4, we may add symbols a and b so that both appear 4 times in P. By Proposition 1 we may further assume that $(1,1,a),(2,1,b) \in P$. Then P contains one of the following arrays, denoted (1), (2), (3), (4), (5), and (6) respectively.

a	
b	b
	a
ab	ab

a	b
b	
	a
ab	ab

a	
b	a
	b
ab	ab

a	b
b	a
ab	ab

a	a
b	b
ab	ab

a	a
b	
	b
ab	ab

Note that (1) and (6), and (2) and (3) are isotopic. So to prove that P is aviodable, we avoid each of (1), (2), (4), and (5). Without loss of generality suppose that $a = 1$ and $b = 2$.

Consider (1) $\subseteq P$. If $(3,2,3),(3,2,4) \notin P$, then we add either symbol 3 or 4 to cell $(3,2)$. Without loss of generality, $(3,2,3) \in P$. (Note that L_1 below contains empty cells. Either $(1,2,3),(2,2,1) \in L_1$ or $(1,2,1),(2,2,3) \in L_1$, depending on where symbol 3 appears in column 2 of P_1.)

1	
2	2
	13
12	12

$\subseteq$

1	
24	2
	13
12	12

$\subseteq$

1	
24	24
	13
12	12

$\subseteq$

1	4
24	24
	13
12	12

$\subseteq$

1	4
24	24
4	13
12	12

2	
4	
1	2
3	4

2	1
3	4
1	2
4	3

2	4
3	1
1	2
4	3

2	1
1	3
4	2
3	4

But then P contains an isotope of Q_1. In this case P is avoidable.

Consider (2) $\subseteq P$. As in the previous case, $(3,2,3) \in P$.

1	2
2	
	13
12	12

$\subseteq$

1	2
2	
3	13
12	12

$\subseteq$

1	2
23	
3	13
12	12

2	1
1	2
3	4
4	3

2	1
3	2
1	4
4	3

3	1
1	2
2	4
4	3

Consider (4) $\subseteq P$. We assume, without loss of generality, that $(1,2,3) \in P$.

$$\begin{array}{|c|c|}\hline 1 & 23 \\ \hline 2 & 1 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 2 & 1 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 2 & 1 \\ \hline 3 & \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 2 & 1 \\ \hline 3 & 3 \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 2 & 1 \\ \hline 3 & 34 \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 2 & 1 \\ \hline 34 & 34 \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 2 & 14 \\ \hline 34 & 34 \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 13 & 23 \\ \hline 24 & 14 \\ \hline 34 & 34 \\ \hline 12 & 12 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 2 \\ \hline 3 & 1 \\ \hline 4 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 1 & 2 \\ \hline 2 & 3 \\ \hline 3 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 2 \\ \hline 1 & 4 \\ \hline 4 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & 3 \\ \hline 4 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 4 \\ \hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & 3 \\ \hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$$

The last partial 2-entry array is an isotope of the row/symbol-conjugate of Q_1. So P, in this case, is avoidable.

Finally consider $(5) \subseteq P$. As in the previous case, $(1,2,3) \in P$.

$$\begin{array}{|c|c|}\hline 1 & 13 \\ \hline 2 & 2 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 1 & 13 \\ \hline 23 & 2 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 14 & 13 \\ \hline 23 & 2 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \quad \textit{or} \quad \begin{array}{|c|c|}\hline 1 & 13 \\ \hline 23 & 23 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \subseteq \begin{array}{|c|c|}\hline 14 & 13 \\ \hline 23 & 24 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array} \quad \textit{or} \quad \begin{array}{|c|c|}\hline 13 & 13 \\ \hline 23 & 23 \\ \hline & \\ \hline 12 & 12 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 1 \\ \hline 1 & 2 \\ \hline 4 & 3 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 4 & 2 \\ \hline 1 & 3 \\ \hline 2 & 1 \\ \hline 3 & 4 \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array} \quad \textit{or} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 & 4 \\ \hline 2 & 1 \\ \hline 4 & 3 \\ \hline\end{array}$$

Thus P is avoidable since $P \notin \mathcal{Q}$. □

Theorem 1 *Let $P \in \mathcal{P}_{4\times 2}$. If $P \notin \mathcal{Q}$, then P is avoidable.*

Proof: We add as many symbols as possible to P. By Lemma 1, if there is a row in P containing two symbols twice, then P is avoidable. So we assume that P contains no such row.

Case 1: There is a pair of symbols $\{a,b\}$ such that both a and b appear 4 times in P and there is exactly one cell containing both a and b.

Without loss of generality we will assume that

$$\{(4,1,a),(4,1,b),(1,1,a),(2,1,b)\} \subseteq P.$$

It follows that P contains one of the following arrays denoted (1), (2), (3), (4), (5), and (6) respectively.

a	b
b	b
	a
ab	a

a	b
b	a
	b
ab	a

a	a
b	b
	b
ab	a

a	b
b	a
	a
ab	b

a	a
b	b
	a
ab	b

a	a
b	a
	b
ab	b

Note that (1) and (6), (2) and (4), and (3) and (5) are isotopic. Without loss of generality, assume that $a = 1$ and $b = 2$ and that $(4, 2, 3) \in P$. Consider $(1) \subseteq P$.

1	2
2	2
	1
12	13

$\subseteq$

1	2
23	2
	1
12	13

2	
3	
1	3
4	2

2	1
4	3
1	2
3	4

Next, consider $(2) \subseteq P$.

1	2
2	1
	2
12	13

$\subseteq$

14	2
2	1
4	2
12	13

4	
1	2
2	
3	4

3	
1	3
2	
4	2

Finally, consider $(3) \subseteq P$.

1	1
2	2
	2
12	13

$\subseteq$

14	1
2	2
	2
12	13

or

1	1
2	23
	2
12	13

$\subseteq$

14	1
2	23
	2
12	13

or

1	14
2	23
	2
12	13

4	2
1	3
2	1
3	4

3	4
1	3
2	1
4	2

or

2	4
3	1
1	3
4	2

2	4
3	1
1	3
4	2

or

4	3
1	4
2	1
3	2

Case 2: For each pair of symbols $\{a, b\}$, a or b appears at most 3 times in P or there are two cells in P containing both a and b.

In this case P can not be completed to a 4×2 2-entry latin rectangle. There is a column, say column 1, and a symbol, say symbol 4, of P such that symbol 4 can not appear twice in column 1. Then column 1 of P is isotopic to the partial 4×1 2-entry latin rectangle in Figure 5. So without loss of generality, we assume that column 1 of P is the array in Figure 5.

12
13
23
4

Figure 5: Column 1 of P.

Of the symbols 1, 2, and 3, two of them must each appear twice in column 2 of P. Without loss of generality, suppose these two symbols are 1 and 2. By Case 1, P contains one of the following arrays denoted (1), (2), and (3) respectively. Note that either $(1, 2, 1) \notin P$ or $(1, 2, 2) \notin P$. We will assume that $(1, 2, 2) \notin P$.

12			12			12	
23	12		23			23	
13			13	12		13	
4			4			4	12

Note that (1) and (2) are isotopic. Consider $(1) \subseteq P$.

12			12			12			12	
23	12		23	12		23	12		23	12
13		$\subseteq$	13	4	$\subseteq$	13	34	$\subseteq$	13	34
4			4			4			4	4

4			4			4	2		3	2
1	3		1	4		1	3		1	4
2	4		2	3		2	1		4	1
3			3			3	4		2	3

And consider $(3) \subseteq P$.

12			12			12			12	23
23			23			23			23	
13		$\subseteq$	13	3	$\subseteq$	13	34	$\subseteq$	13	34
4	12		4	12		4	12		4	12

3	
4	
2	3
1	4

3	
4	
2	4
1	3

4	
1	
2	1
3	4

3	4
4	2
2	1
1	3

□

Consider the arrays Q_1 and Q_2 in Figure 1. The 4×4 arrays $Q_1^* = \{(i,k,j) : (i,j,k) \in Q_1\}$ and $Q_2^* = \{(i,k,j) : (i,j,k) \in Q_2\}$ are given in Figure 6. Because Q_1 and Q_2 are unavoidable, Q_1^* and Q_2^* are unavoidable. We use $\mathcal{Q}^*$ to denote the set

$$\{P \in \mathcal{P}_4 : \; P \text{ contains an isotope of } Q_1^* \text{ or } Q_2^*\}$$

	2	1	
	1	2	
12		12	
12	12		

	12	12	
12		12	
12	12		

Figure 6: Arrays Q_1^* and Q_2^* respectively

Corollary 1 *The set $\mathcal{Q}^*$ contains all the unavoidable partial 2-entry latin squares of order 4 on the symbol set $\{1,2\}$.*

References

[1] Casselgren, C.J.: On avoiding some families of arrays. Discrete Math. 312, 963-972 (2012)

[2] Cavenagh, N.: Avoidable partial latin squares of order $4m+1$. Ars Combin. 95, 257-275 (2010)

[3] Chetwynd, A.G., Rhodes, S.J.: Avoiding multiple entry arrays. J. Graph Theory 25, 257-266 (1997)

[4] Chetwynd, A.G., Rhodes, S.J.: Avoiding partial Latin squares and intricacy. Discrete Math. 177, 17-32 (1997)

[5] Öhman, L.-D.: Partial latin squares are avoidable. Ann. Comb. 15, 485-497 (2011)