A Brief Survey of Elliptic Geometry

By

Justine May

Advisor: Dr. Loribeth Alvin

An Undergraduate Proseminar
In Partial Fulfillment of the Degree of
Bachelor of Science in Mathematical Sciences
The University of West Florida
April 2012

<u>APPROVAL PAGE</u>

The Proseminar of Justine May is approved:

_____   _____
Lori Alvin, Ph.D., Proseminar Advisor   Date

_____   _____
Josaphat Uvah, Ph.D., Committee Chair   Date

Accepted for the Department:

_____   _____
Jaromy Kuhl, Ph.D., Chair   Date

Abstract

There are three fundamental branches of geometry: Euclidean, hyperbolic and elliptic, each characterized by its postulate concerning parallelism. Euclidean and hyperbolic geometries adhere to the all of axioms of neutral geometry and, additionally, each adheres to its own parallel postulate. Elliptic geometry is distinguished by its departure from the axioms that define neutral geometry and its own unique parallel postulate. We survey the distinctive rules that govern elliptic geometry, and some of the related consequences.

Table of Contents

Chapter 1: Introduction

I.   Problem Statement

In Euclidean geometry, that is the most familiar geometry to the majority of people, Euclid's fifth postulate is often stated as "For every line *l* and every point P that does not lie on *l,* there exists a unique line *m* through P that is parallel to *l*" [Greenberg 21].   Euclid's fifth postulate has proven controversial throughout the study of Geometry, as many mathematicians maintained that this seemingly "obvious" postulate could be derived from the other four.   These four additional postulates first posed by Euclid are the following:

1. For every point P and every point Q not equal to P there exists a unique line that passes through P and Q.
2. For every segment AB and for every segment CD there exists a unique point E on the line $\overleftrightarrow{AB}$ such that B is between A and E and segment CD is congruent to segment BE.
3. For every point O and every point A not equal to O, there exists a circle with center O and radius OA.
4. All right angles are congruent to each other.

Many famous mathematicians, including Johann Heinrich Lambert and Giovanni Gerolamo Saccheri, attempted to prove that Euclid's fifth postulate was a necessary result of the other four postulates, but every attempt that was made proved unsuccessful.

Though Euclid's *Elements* was one of the first mathematical publications to amass all to-date known geometry into one work with a more rigorous treatment than was commonly practiced prior, improvements could still be made. David Hilbert, in his work *Grundlagen der Geometrie* in 1899, modified and specified Euclid's system of axioms while providing an even more rigorous treatment than that first presented by Euclid [Greenberg 104].

Euclid's first four postulates comprise what is now known as neutral geometry; a geometry without assumption of a parallel postulate. However, as all-encompassing as neutral geometry may seem, there still exist valid geometric models for which the maxims of neutral geometry are insufficient. The world in which we live exists on the surface of a sphere and does not adhere to even these four simple postulates. In order to describe and understand this system, we must redirect the focus to a very different type of geometry called elliptic geometry.

II.   Relevance

Elliptic geometry is a geometry in which no parallel lines exist. In order to discuss the rigorous mathematics behind elliptic geometry, we must explore a consistent model for the geometry and discuss how the postulates posed by Euclid and amended by Hilbert must be adapted. The most common and intuitive model of elliptic geometry is the surface of a sphere. In the spherical model a "point" is defined as a pair of antipodal points and a "line" is defined as a great circle of the sphere.

This model may sound deceptively simple at first glance, but since this model has no parallel lines, it is not a model of neutral geometry. See Chapter 2 Section I for an overview of the axioms of neutral geometry. In Chapter 2, we will redefine terms and accept the idea that parallel lines do not exist in our geometry. Then we will modify some of the axioms familiar to neutral geometers.

In order to understand elliptic geometry, we must first distinguish the defining characteristics of neutral geometry and then establish how elliptic geometry differs. A Euclidean geometric plane (that is, the Cartesian plane) is a sub-type of neutral plane geometry, with the added Euclidean parallel postulate. Hyperbolic geometry is another sub-type of neutral plane geometry with the added hyperbolic parallel postulate, which states that through any point P not on a line l, there exist multiple lines m parallel to l.

Neutral Geometry is comprised of David Hilbert's 13 main Axioms (3 incidence axioms, 4 betweenness axioms, and 6 congruence axioms) and several additional continuity axioms. The incidence axioms give a more thorough explanation of Euclid's first postulate, whereas the betweenness and congruence axioms are necessary to more rigorously modify Euclid's second postulate. We note that these new axioms can be used to prove Euclid's fourth postulate, making that postulate obsolete; the additional axioms of continuity are used to replace the third postulate. Neutral geometry is notably without a postulate or axiom concerning parallel lines. This isn't to say that parallel lines do not exist in neutral geometry (in fact, they do). It is to say that in any neutral geometric model, we do not commit to *how many* parallel lines exist through a given point not on the line. In Euclidean geometry, such a line would be unique, whereas hyperbolic geometry allows for infinitely many such lines [Greenberg, 75].

Many of the axioms of neutral geometry (in particular the incidence axioms and most of the congruence axioms) are still valid in elliptic geometry. However, the betweenness axioms must be redefined entirely in order to be useful, as "betweenness" is not a valid concept in elliptic geometry. When dealing with a circle, how do we define which point is between the other two?

As we explore Hilbert's axioms of neutral plane geometry, we will note if and how the axioms must be adjusted to fit the elliptic model.

III.   Literature Review

A large portion of this paper, including the discussion of the axioms of neutral geometry and the axioms of elliptic geometry, relies heavily on the text *Euclidean and Non-Euclidean Geometries* by Greenberg [2]. Another noteworthy text in the study of geometry is *Non-Euclidean Geometry* by Coxeter [1]; this text was primarily used to gain additional perspectives on ideas presented by Greenberg. One of the first treatments of geometry on the sphere was given by Wilson in [9]; his series of articles gave validity to the study of elliptic geometry on the sphere.

In order to better understand the intricacies of elliptic geometry, it is helpful to have a practical model to manipulate; The *Spherical Geometry Explorer* at David Little's website [4] provides an excellent environment to explore; this applet is also discussed in [3]. For a cursory introduction to Girard's formula for spherical excess, see [7]. Finally, although many of the graphics in this paper were constructed using Microsoft Paint, the remaining sources provided inspiration and foundation for those figures.

Chapter 2: Main Body

I.   The Axioms of Neutral and Elliptic Geometries

In this section, we discuss the axioms or postulates behind the major areas of geometry, namely the axioms of neutral geometry and the axioms of elliptic geometry

1. Axioms of Incidence

In this section we now discuss the incidence axioms. These axioms were originally stated to clarify and remove any ambiguity from Euclid's first postulate. In elliptic geometry, and in particular on the spherical model, because we must reinterpret the terms "point" and "line", it is advantageous to rephrase each axiom. We use Figure 1 to help interpret the axioms.
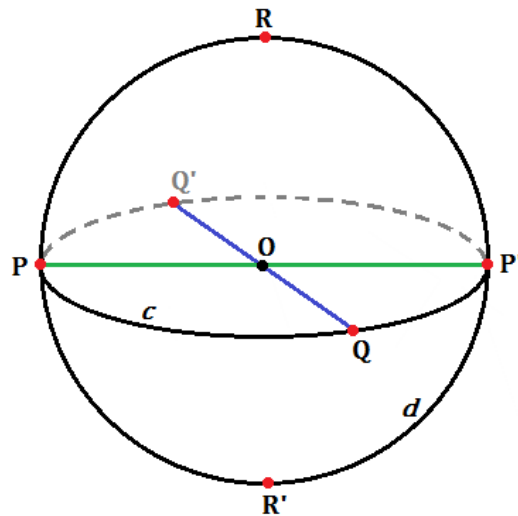
Figure 1 **A visual representing the Incidence Axioms.**

Incidence Axiom 1: *For every point P and for every point Q (Q ≠ P) there exists a unique line l incident with P and Q.*

Recall that in elliptic geometry we define points as pairs of antipodal points, and lines as great circles on a sphere. Hence we may reword this axiom in the following way:

For every pair of antipodal points P and P' and for every pair of antipodal points Q and Q' such that Q ≠ P(and thus Q'≠P') there exists a unique great circle incident with both pairs of antipodal points.

Although a circle requires only three points to be defined, examination of the relationship of Q' to Q indicates that the circle defined only by P, P', and Q would inevitably travel through Q' as well. Likewise, though two points do not define a unique circle in the plane, as we are limiting "lines" to mean *great* circles on the sphere, two distinct points define a unique circle.

Incidence Axiom2: *For every line l, there exist at least two distinct points incident with l.*

This axiom may be reworded in the following way for our spherical model of elliptic geometry.

> *For every great circle c, there exist at least two distinct pairs of antipodal points incident with c.*

Incidence Axiom 3: *There exist three distinct points with the property that no line is incident with all three of them.*

Once again, with our reinterpretation of a point, we may rephrase this axiom as follows.

> *There exist three distinct pairs of antipodal points with the property that no great circle is incident with all three of them.*

In Figure 1, *c* is incident with P,P', Q and Q', and *d* is incident with P, P', R and R'. No great circle exists which is incident with all three pairs of antipodal points. That is, the spherical model satisfies the incidence axioms.

2. Axioms of Betweenness

The axioms of betweenness and congruence are used to clarify and remove ambiguity from Euclid's second postulate. In this section we discuss the betweenness axioms for neutral geometry and note that they will be replaced by the *separation axioms* of elliptic geometry; See Section II.3.

Defining the notation for betweenness in the following way,

$$A * B * C \equiv \text{point B is between points A and C,}$$

there are four axioms of betweenness from Hilbert, which are given below.

Betweenness Axiom 1: *If A*B*C then A, B, and C are three distinct points all lying on the same line and C*B*A.*

Betweenness Axiom 2: *Given any two distinct points B and D, there exist points A, C, and E lying on $\overleftrightarrow{BD}$ such that A*B*D, B*C*D, abd B*D*E*

Betweenness Axiom 3: *If A, B, and C are three distinct points lying on the same line, then one and only one of the points is between the other.*

Betweennes Axiom 4: *(Plane Separation) For every line l and for any three points A, B, and C not lying on l:*

      (i)   *If A and B are on the same side of l and B and C are on the same side of l, then A and C are on the same side of l.*

      (ii)   *If A and B are on opposite sides of l and if B and C are on opposite sides of l, then A and C are on the same side of l*

As mentioned earlier, the axioms of betweenness are not valid when "lines" are redefined to be circles. It may not be immediately apparent that betweenness is not a relevant concept on a circle. For example, does the assignment of betweenness differ when traveling clockwise around the circle versus counter-clockwise?

In the following illustration, it seems intuitive in both cases that point B should be between points A and C.
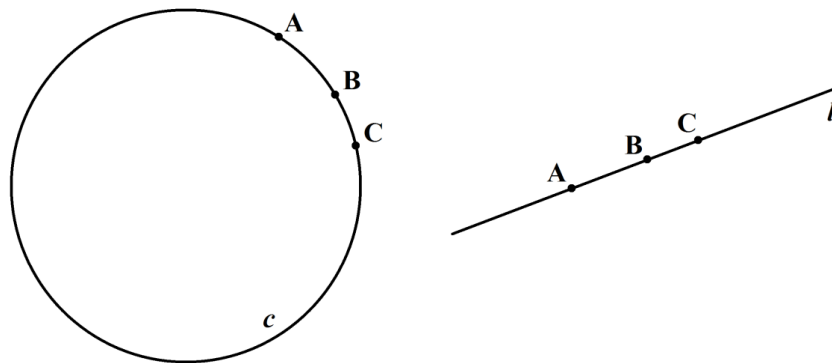
Figure 2 **Exploring betweenness in neutral and elliptic geometry.**

Whereas betweenness is well defined for equidistant points in neutral geometry, in elliptic geometry the assignment of betweenness is arbitrary at best. This is portrayed in Figure 3.
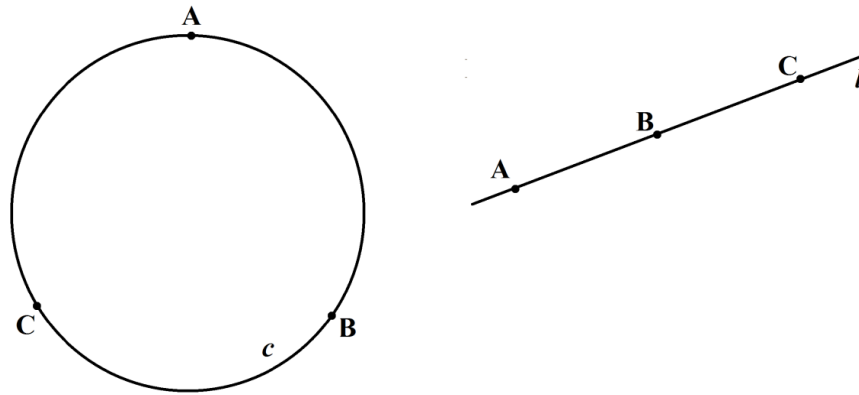
Figure 3 **An intuitive understanding of betweenness fails on a circle.**

Since betweenness is ill-defined within the parameters of elliptic geometry, but access to the concepts which the betweenness axioms of neutral geometry define is required, elliptic geometry exchanges the relation and axioms of betweenness for those of "separation."

3. Axioms of Separation

In this section we explore the concept of separation. We define the notation of the separational relations in the following way:

$$(A, B | C, D) \equiv \textit{"points A and B separate points C and D"}$$

It may also be stated for clarity that when traversing the circle from point C to point D either point A or point B will be "crossed over" in the process, which we will see more clearly in the formal statement of the separation axioms.
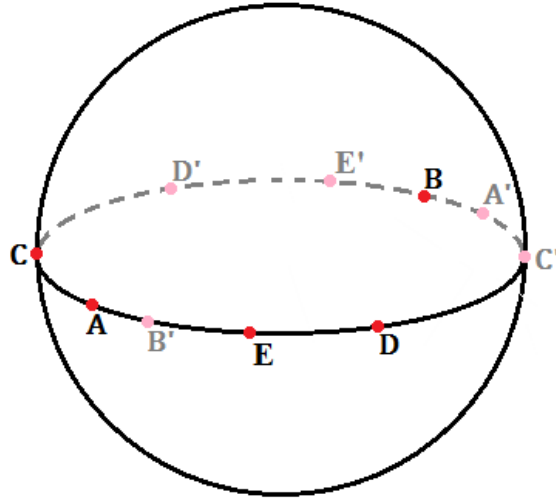
Figure 4 An visual explanation of separation axioms 1-3, 5, and 6.

Separation Axiom 1: If $(A, B | C, D)$, then points A, B, C, and D are collinear and distinct.

In other words, non-collinear points cannot separate one another.

Separation Axiom 2: If $(A, B | C, D)$, then we have $(C, D | A, B)$ and $(B, A | C, D)$.

We observe that separation is reflexive and symmetric. If you cannot get from C to D without passing A or B, then to get from A to B, C or D must be crossed. Also, if you cannot get from C to D without crossing A or B, then you cannot get from C to D without crossing B or A.

Separation Axiom 3: If $(A, B | C, D)$, then not $(A, C | B, D)$.

The emphasis here is that separation is well-defined.

Separation Axiom 4: If points A, B, C, and D are collinear and distinct then $(A, B | C, D)$ or $(A, C | B, D)$ or $(A, D | B, C)$. That is, one (and only one) of the following three graphics is valid (See Figure 5.)
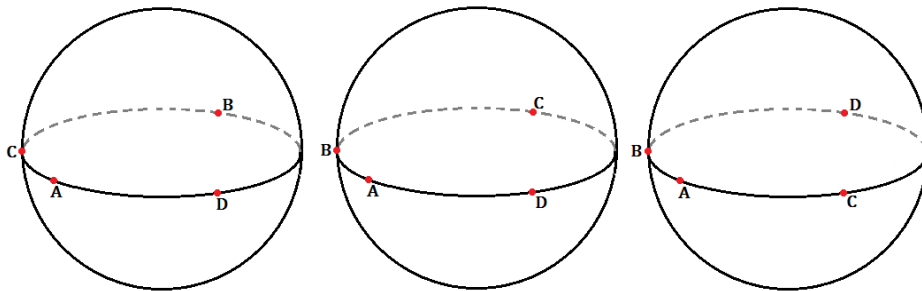
**Figure 5** A visual representation of Separation Axiom 4.

8

Separation Axiom 5: *If points A, B, and C are collinear and distinct then there exists a point D such that* $(A, B|C, D)$.

In other words, given three points a fourth point may always be found such that the first two points separate the third point and the new point.

Separation Axiom 6: *For any five distinct collinear points, A, B, C, D, and E, if* $(A, B|D, E)$, *then either* $(A, B|C, D)$ *or* $(A, B|C, E)$.

The following definition and diagram help us to better understand separation axiom 7.

Perspectivity: Let *l* and *m* be any two lines and let O be a point not on either of them. For each point A on line *l*, the line $\overleftrightarrow{OA}$ intersects *m* in a unique point A'' (recall the elliptic parallel property). The one-to-one correspondence that assigns A'' to A for each A on *l* is called the *perspectivity* from *l* to *m* with center O. For ease, the picture below is drawn projected onto the plane. See Figure 6.
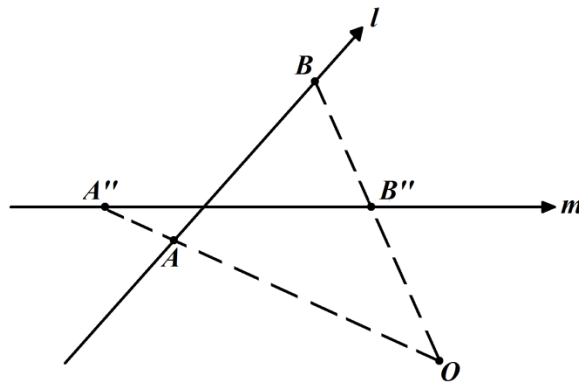
Figure 6 **Perspectivity with center point O.**

*Separation Axiom 7:* *Perspectivities preserve separation; i.e., if* $(A, B|C, D)$, *with l the line through A, B, C, and D, and A'', B'', C'', and D'' are the corresponding points on line m under a perspectivity, then* $(A'', B''|C'', D'')$.

The separation axioms provide a manner of interpreting the betweenness axioms that remains valid on a circle. These axioms are used in elliptic geometry in much the same way that the betweenness axioms are used in neutral geometry.

4. Axioms of Congruence

The definition of "segment" needs to be restated for elliptic geometry, as the neutral definition uses the concept of betweenness.

A *segment* (or arc) AB in elliptic geometry is ambiguous when defined as the set of all points, on the great circle, lying between A and B, as there are two segments available which meet those criteria since betweenness is ill-defined on circles. However, if we redefine a segment using the concept of separation, rather than betweenness, we are able to be explicit.
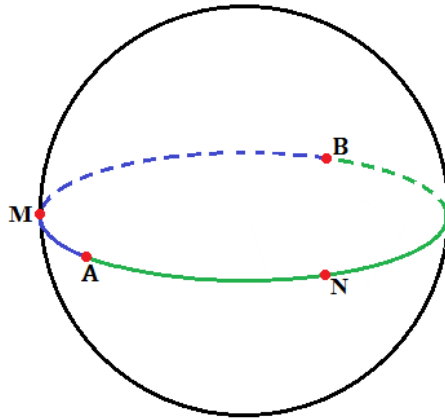
Figure 7 **An illustration of the two line[AB]ₙ [AB]ₘ.**

When defining a segment of a great circle we may use the notation $[AB]_N$ to denote the set of all points X that lie on AB such that N does not separate any point X from A and B. To put it more simply: $[AB]_N$ denotes the segment AB which is not incident with N (depicted in Figure 7 in blue). Similarly, $[AB]_M$ denotes the segment not incident with M (depicted in Figure 7 in green).

When A and B form non-congruent arcs (that is $B \neq A'$, or B is not antipodal to A) then one arc will span a length less than πr (where r is the radius of the sphere) and the other arc will span a length greater than πr; noting that an entire line would have length $2\pi r$. In this case the arcs may be distinguished as the major ($AB > \pi r$) and minor ($AB < \pi r$) arcs of the great circle defined by points A and B.

Below are the congruence axioms as stated by Hilbert in neutral geometry.

Congruence Axiom1: *If A and B are distinct points and if A' is any point, then for each ray r emanating from A' there is a UNIQUE point B' on r such that $B' \neq A'$ and $AB \cong A'B'$.*

Congruence Axiom2: If $AB \cong CD$ and if $AB \cong EF$, then $CD \cong EF$. Moreover, every segment is congruent to itself.

Congruence Axiom 3: If $A*B*C, A'*B'*C', AB \cong A'B'$, and $BC \cong B'C'$, then $AC \cong A'C'$.

Congruence Axiom 4: Given any $\sphericalangle BAC$ (where, by the definition of "angle," $\overrightarrow{AB}$ is not opposed to $\overrightarrow{AC}$) and given any ray $\overrightarrow{A'B'}$ emanating from a point A', then there is a UNIQUE ray $\overrightarrow{A'C'}$ on a given side of line $\overleftrightarrow{A'B'}$ such that $\sphericalangle B'A'C' \cong \sphericalangle BAC$.

Congruence Axiom 5: if $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$ then $\sphericalangle B \cong \sphericalangle C$. Moreover, every angle is congruent to itself.

Congruence Axiom 6: (SAS) If two sides and the included angle of one triangle are congruent, respectively, to two sides and the included angle of another triangle, then the two triangles are congruent.

These six congruence axioms may be reworded in the following way to form a set of equivalent congruence relations in the language of elliptic geometry. Refer to Figure 8 for congruence axioms 1-7, as interpreted on the sphere. Note that using the language of elliptic geometry, it is simpler and more explicit to restate these six axioms into eight equivalent axioms.
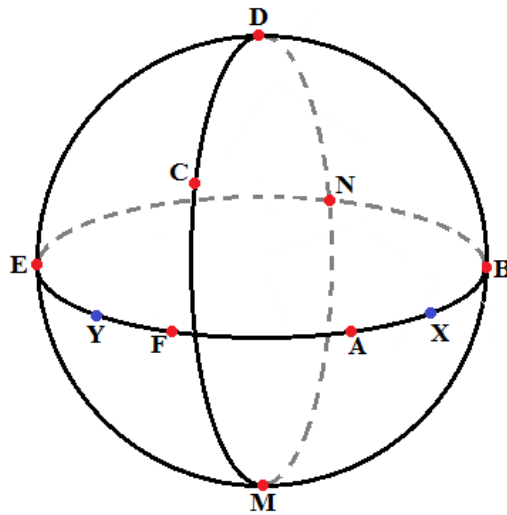

Figure 8 Visual for Congruence Axioms 1-7.

Congruence Axiom1: Each segment is congruent to itself.

Congruence Axiom2: If $[AB]_N \cong [CD]_N$ then $[CD]_N \cong [AB]_N$

11

Congruence Axiom 3: If $[AB]_N \cong [CD]_N$ and $[CD]_N \cong [DF]_N$ then $[AB]_N \cong [EF]_N$

Congruence Axiom 4: If $[AB]_N \cong [CD]_N$ then $[AB]_M \cong [CD]_M$

Congruence Axiom 5: If $[AB]_N \cong [AB]_M$ then A is antipodal to B.

Congruence Axiom 6: If A and A' are antipodal points then both of the segments AA' are congruent.

Congruence Axiom 7: If $[AB]_N \cong [EF]_N$ and there exists a point X on $[AB]_N$ then there exists a point Y on $[EF]_N$ such that $[XB]_N \cong [YF]_N$

Congruence Axiom 8: (SAS) If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent

That is, if $[AB]_N \cong [CD]_N$ and $[BX]_D \cong [DX]_B$ and ∢ABX≅ ∢CDX then ΔABX ≅ ΔCDX (see Figure 9).
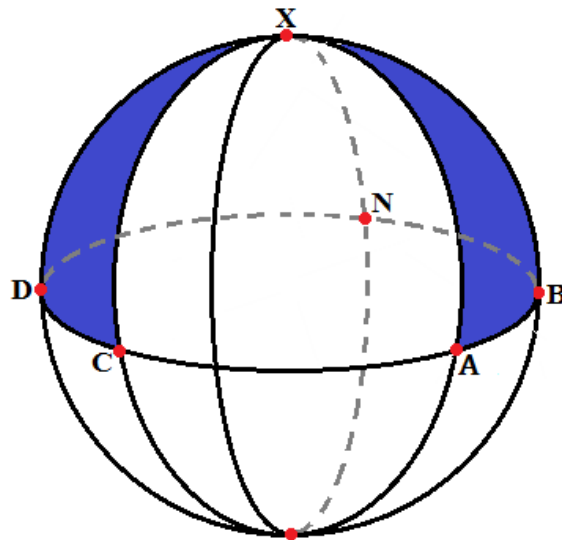
Figure 9 **Representation of SAS on the sphere.**

Now that we have modified each of the necessary axioms, we now begin exploring elliptic polygons on the sphere.

II.   Lunes (Biangles)

Unlike in Euclidean geometry, it is possible to create a two-sided polygon on the elliptic sphere; these biangles are commonly known as *lunes*. A lune is defined by the intersection of two great circles and is determined by the angles formed at the antipodal points located at the intersection of the two great circles, which form the vertices of the two angles. See Figure 10.
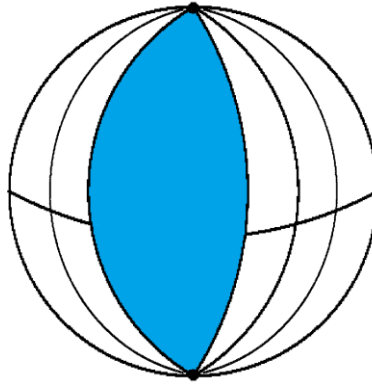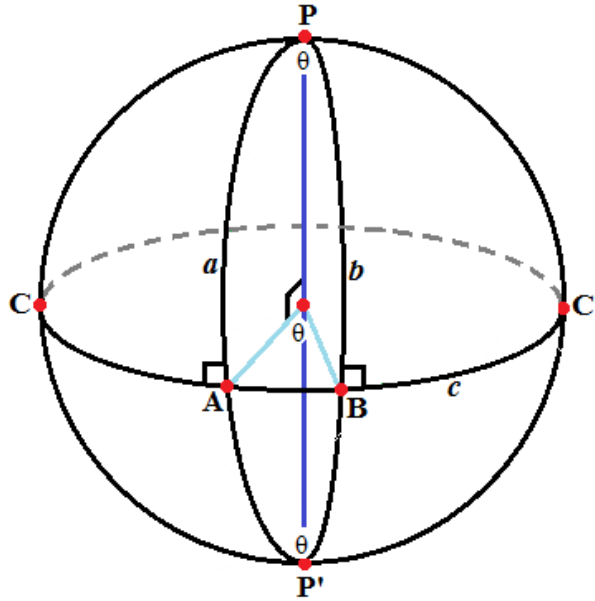

Figure 10 **A lune.**

To measure an angle on a sphere, we must use the tangent plane to the sphere at the vertex of the angle. Consider the two lines on this tangent plane that when projected onto the sphere align onto the two great circles forming the angle. We define the angle between two great circles to be the angle between the associated lines on the tangent plane. This angle is also congruent to the angle emanating from the origin of the sphere to the intersection of the sides of the lune and the great circle orthogonal to the vertices of the lune, which we refer to as the orthogonal great circle of the lune. This angle is called the *lunar angle* and is labeled θ in Figure 11; c is the orthogonal great circle.

13

Given a lune, we may calculate its area and the length of its lunar arc. Note that the total surface area of the sphere is $4\pi r^2$. Multiplying the total surface area by the ratio of the angle θ to the total 2π radians in a circle will give the area of the lune

$$A_L = 2\theta r^2.$$

The length of the lunar arc on the orthogonal great circle (segment [AB]c in Figure XXX) is given by multiplying the total length of the great circle *c*, which is $2\pi r$, by the ratio of the angle θ to the total 2π radians in a circle. Thus the length of the lunar arc is

$$L = \theta r.$$

We now use our knowledge about lunes to study another family of polygons on the sphere: triangles.

14

III.   Triangles

Triangles on the sphere are defined similarly to Euclidean triangles: they consist of three vertices and the arcs of great circles that join these vertices, which are called the sides. The area of the triangle is the surface area of the region of the sphere enclosed by the sides. However, when a triangle is constructed on a sphere, there is room for a great deal of ambiguity. There are two segments that connect each pair of points on a great circle, the major and a minor arc. Once a decision has been reached concerning which of these two segments to use as a side of the triangle, the resulting triangle has no clear interior or exterior. That is, the sides of the triangle form the boundary for two different regions. For "small" triangles on "large" spheres there is an intuitive choice as to what region forms the interior of the triangle, namely the smaller region formed by the three sides, but in order to be precise we need to clearly establish which region we select in order to be precise. Furthermore, when the triangles become larger or the elliptic sphere becomes smaller, even intuition may not provide much insight into which region to consider as the triangle formed by the three given points.

Given any three points on a sphere, there are eight possible triangles to be considered. Each is technically a valid triangle in elliptic geometry. Intuitively, it makes sense to choose the sides of the triangle to always be formed by minor arcs when possible. For ease, we begin our examples and theorems by choosing this intuitive triangle and then discuss the other possibilities.

Exploring these complexities of spherical triangles allows us to understand elliptic geometry more fully. It is notable that all triangles on the sphere have an angle sum which is greater than π, whereas triangles in neutral geometries have angle sums either exactly π (in Euclidean space) or less than π (in the hyperbolic plane). Another interesting consequence is that the angle sum in excess of π resulting from the construction of a spherical triangle is related to the area of that triangle; this effect is known as Girard's theorem.

IV.   Girard's Theorem

For most geometrical constructs on the sphere, Girard's Theorem plays a large role. The following section provides a discussion of how Girard's Theorem applies to triangles.

Girard's Theorem:

$$\sum interior\ angles\ of\ triangle\ on\ a\ sphere = \frac{area\ of\ the\ triangle}{radius\ of\ the\ sphere} + \pi.$$

Proof:

Let S be a sphere and let T be a triangle on S with vertices A, B and C with interior angles α, β, and γ, respectively. See Figure 12(a). We note that in the following α, β, and γ, represent both the actual angle and the measure of the angle depending upon the context. Let T' be the antipodal triangle to T. It follows that the vertices of T' are A', B' and C' (antipodal points to A, B, and C, respectively); see Figure 12(b).
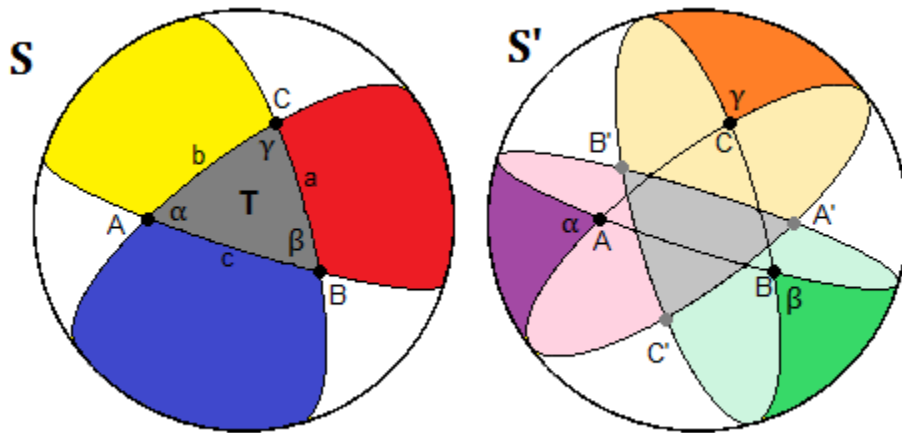

Figure 12(a) Spherical triangle T.   **Figure 12(b)** Antipodal triangle T'.

Note that the lines b and c form angle α at both A and A', defining a pair of congruent lunes (one lune containing T and one containing T'). In Figure 12(a) the lune formed from α at A is colored red, whereas the lune formed from α at A' is colored purple in Figure 12(b). Interestingly, this is a result of the vertical angle theorem, which still holds in elliptic geometry. Let the lune with vertices A and A' containing T be called L_α and the one containing T' be called L'_α.

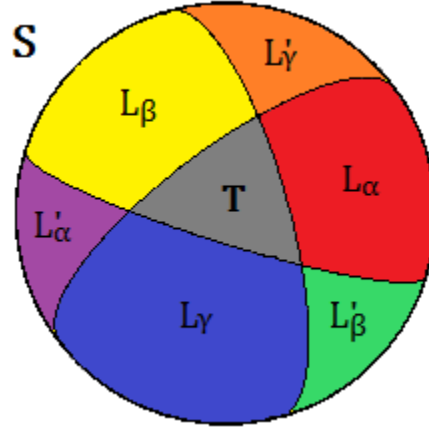The area of the congruent lunes is given by

$$A(L_\alpha) = A(L'_\alpha) = 2(\alpha)r^2.$$

Similarly,

$$A(L_\beta) = A(L'_\beta) = 2(\beta)r^2 \text{ and}$$
$$A(L_\gamma) = A(L'_\gamma) = 2(\gamma)r^2.$$

See Figure 13.

Figure 13 All six lunes containting T and T' on the sphere.

Since each of the six lunes defined above contain either T or T', the sum of the areas of all six lunes accounts for the entire sphere plus an additional 2 copies of T and 2 copies of T'. This is represented by

$$A(L_\alpha) + A(L'_\alpha) + A(L_\beta) + A(L'_\beta) + A(L_\gamma) + A(L'_\gamma) = 4\pi r^2 + 2A(T) + 2A(T'),$$

where A(T) represents the area of T. Since $A(T) = A(T')$, it follows that

$$A(L_\alpha) + A(L'_\alpha) + A(L_\beta) + A(L'_\beta) + A(L_\gamma) + A(L'_\gamma) = 4\pi r^2 + 4A(T)$$

Since $A(L_\alpha) = A(L'_\alpha)$, $A(L_\beta) = A(L'_\beta)$, and $A(L_\gamma) = A(L'_\gamma)$, we may again simplify our equation to obtain

$$2A(L_\alpha) + 2A(L_\beta) + 2A(L_\gamma) = 4\pi r^2 + 4A(T).$$

As the area of a lune of angle θ is given by $2(\theta)r^2$,

$$2(2(\alpha)r^2) + 2(2(\beta)r^2) + 2(2(\gamma)r^2) = 4\pi r^2 + 4A(T).$$

Therefore we have

$$4(\alpha)r^2 + 4(\beta)r^2 + 4(\gamma)r^2 - 4\pi r^2 = 4A(T)$$

$$(\alpha)r^2 + (\beta)r^2 + (\gamma)r^2 - \pi r^2 = A(T)$$

$$(\alpha) + (\beta) + (\gamma) - \pi = \frac{A(T)}{r^2}.$$

Assuming without loss of generality that we are on the unit sphere ($r = 1$), we obtain

$$(\alpha) + (\beta) + (\gamma) - \pi = A(T).$$

■

V.   Consequences of Girard's Theorem

In this section we discuss several consequences of Girard's Theorem, including distortion of maps, similarity of triangles, congruence of triangles, the Pythagorean Theorem, and relative size of triangles on spheres.

1. Maps

Since we live on a sphere but sometimes find it practical to carry a flat map rather than a globe, it is important to understand types of planar projections of the sphere and their limitations. Mapping the surface of a sphere (or even a portion of a spherical object) to the plane involves distortion.

Note that a stereographic projection is by definition smooth, conformal, and bijective. A conformal projection is angle preserving, though distances may exhibit distortion. An isometric projection preserves distances, but angles may show some distortion.

An ideal map is stereographic *and* it would map great circles to straight lines. Girard's theorem lends a decisive argument regarding the existence of ideal maps. Mapping great circles to straight lines would require that a spherical triangle map to a planar triangle, but as the angle sum would change during this process (recall that triangles in the Euclidean plane have angle sum π, whereas spherical triangles have a larger angle sum), the map would not be conformal. Likewise, if angle measure were maintained, the planar triangle would necessarily have curved sides. Thus the conformal map would not be able to convert great circles to straight lines, and an ideal map does not exist.

The Mercator projection is conformal but does not map great circles to straight lines in the plane. The preservation of angles allows navigators (who prefer constant compass direction) to follow rhumb lines, which intersect the meridians of longitude at constant angle. These lines appear as straight lines on a Mercator projection.

A gnomic projection maps great circles to straight lines and is formed by projection from the center of the sphere onto a plane tangent to the sphere. Angles are not preserved in gnomonic projections, but these projections are useful to pilots, since great circles are mapped to straight lines. As we are taught that the shortest distance between two points is a straight line, the gnomonic projection appeals to the desire to preserve that concept. Though the shortest distance between two points on

19

the sphere is the minor arc of a great circle, the mapping of great circles to straight lines provides an intuitive visual guide. See Figure 14 for examples of both the Mercator and gnomic projections.

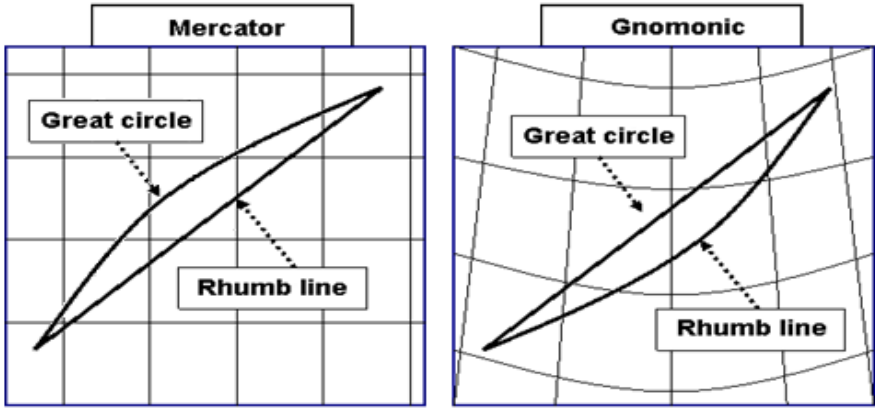## 2. Similarity

Similar triangles (that is, triangles with equal angle measures but different areas) do not exist in elliptic geometry. The area of a spherical triangle is correlated to the sum of the interior angles of the triangle. On any given sphere, S, triangles T and T' (with congruent interior angles,$\alpha, \beta,$ and γ) will necessarily be congruent. This is a direct consequence of Girard's Theorem.

Girard's Theorem states that on the unit sphere, the area of a triangle is equal to portion of the angle sum in excess of π. Hence, any triangle on the unit sphere with angles A, B and C will have an area of $A + B + C - \pi = A(\Delta_{ABC})$. By having the same angle measure, the triangles must be similar, but since the area is also the same, the two spherical triangles are not just similar, but congruent. We now discuss congruence criteria for triangles.

3. Corresponding Parts of Congruent Triangles are Congruent

The congruence criteria for triangles in elliptic geometry differ from those in neutral geometry. Each of the known congruence rules that are assumed in Euclidean and hyperbolic geometries must be re-evaluated in the elliptic case.

The valid congruence theorems for triangles in neutral geometry, namely SSS, SAS, AAS, and ASA (where A stands for angle and S stands for side), must be re-evaluated, and we must add AAA to our congruence criteria for triangles on the sphere. A discussion of each case follows, below.

Congruence Axiom 6 (Side-Angle-Side), as discussed when investigating the congruence axioms, is still considered valid in elliptic geometry. This axiom states that if two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the two triangles are congruent.

Angle-Angle-Angle is the criterion for *similar* triangles in neutral geometry. A direct result of Girard's theorem is that in elliptic geometry, all similar triangles are congruent. Hence we may add angle-angle-angle as a *congruence* criterion in this geometry.
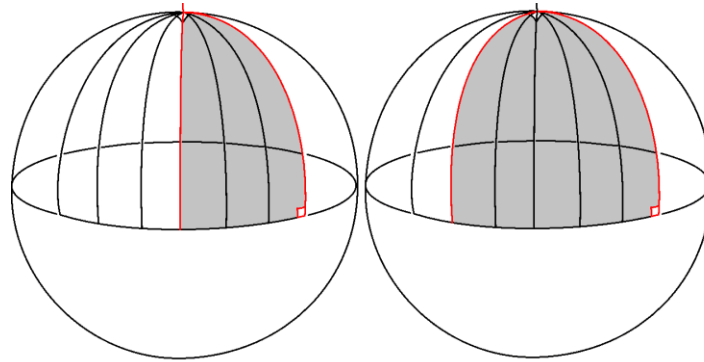
Angle-Side-Angle also holds in elliptic geometry. This criterion leaves no ambiguity regarding the lengths of the unspecified sides or the measure of the third angle. Since the length of the included side defines the unspecified angle, we have defined all three angles and hence prove congruence based on the angle-angle-angle criterion.

Angle-Angle-Side is not valid in elliptic geometry. Though angle-angle-side is a sufficient criterion for congruence in neutral geometry, it is not sufficient in elliptic geometry as all the arcs extending from one pole to its equator, meet the equator at right angles. Hence, any number of triangles may have two angles and a non-included side in common while the third angle varies greatly, disproving congruence. See Figure 15.
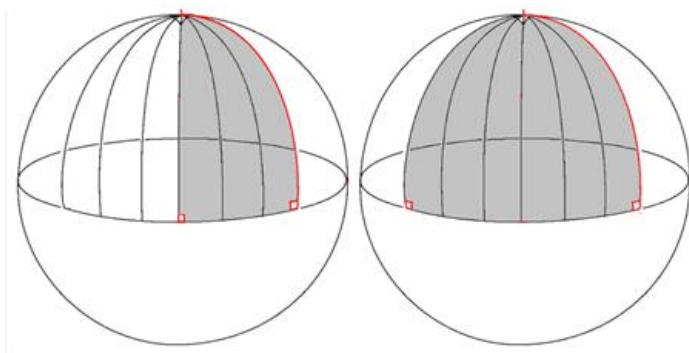
Figure 15 Failure of Angle-Angle-Side

Angle-Side-Side is not a valid criterion in elliptic geometry nor neutral geometry. Since the arc length of the third side is entirely dependent upon the (unspecified) opposite angle, there are several non-congruent constructions that meet this criterion. Hence, the criterion is insufficient to prove congruence. See Figure 16.

Figure 16 **Failure of Side-Side-Angle**

Side-Side-Side is valid in elliptic geometric as in neutral geometry, since any sphere has constant curvature. Girard's theorem proves that all similar triangles are congruent on the sphere, and since the length of each side, $L= \theta r$ (where θ equals the measure of the angle opposite the arc), forces two congruent sides to induce congruent interior angles, we will then ensure congruent triangles by the angle-angle-angle criterion.
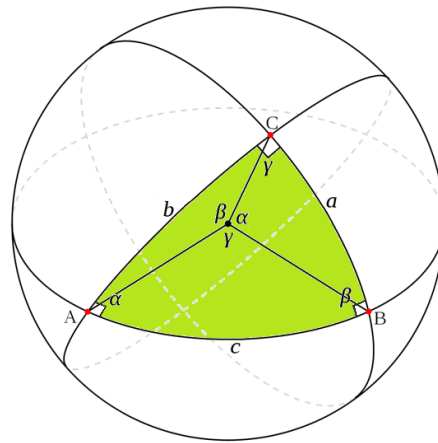
4. The Pythagorean Theorem

Theorem: The Pythagorean Theorem is not valid in elliptic geometry.

Proof:

Assume the Pythagorean Theorem is valid in elliptic geometry.

Consider the spherical triangle T with vertices A, B and C and three right angles: $\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}, \gamma = \frac{\pi}{2}$, and sides a, b, and c, as given in Figure 17.

Figure 17 A triangle with three right angles.

Since the arc length of each side is $L = \theta r$, where θ is the angle from the origin to each endpoint of the arc,

$$L_a = \alpha r = \frac{\pi}{2}$$
$$L_b = \beta r = \frac{\pi}{2}$$
$$L_c = \gamma r = \frac{\pi}{2},$$

where L_a represents the length of side a.

Without loss of generality, let a and b be considered the sides of the right triangle and c be the hypotenuse.

Then we have the following collection of equivalent equations,

$$a^2 + b^2 = c^2$$

$$\left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{2}\right)^2 = \left(\frac{\pi}{2}\right)^2$$

$$\frac{\pi^2}{2} = \frac{\pi^2}{4},$$

a contradiction. Thus, the Pythagorean Theorem is not a valid theorem in elliptic geometry. ∎

Note that as seen in the proof the failure of the Pythagorean Theorem in elliptic geometry, triangles with three right angles do exist.

5. Relative Size of Triangles on the Sphere

As Girard's theorem states,

$$A(\triangle) = r^2(\alpha + \beta + \gamma - \pi)$$

Consider small triangles on large spheres. That is, the area of a triangle is approaching 0, or the radius of the sphere approaches infinity.

$$A(\triangle) \to 0, or \; r^2 \to \infty$$

Hence,

$$\sum (Interior\; angles) = \pi + \frac{A(\triangle)}{r^2}$$

For example, let the radius of the sphere be 1000 units, and the area of the triangle be 0.1 units^2.

Then we have:

$$\sum (Interior\; angles) = \pi + \frac{0.1}{1000^2} = \pi + \frac{0.1}{1000000} = \pi + 0.0000001$$

The sum of the angles is only a very small amount of radians greater than π. Clearly, as r increases or as $A(\triangle)$ decreases, the second fraction will approach 0.
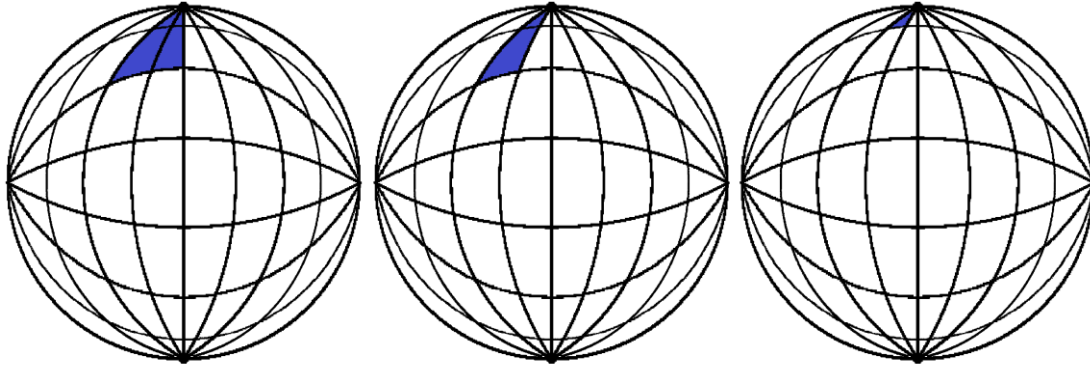
Figure 18 A small triangle on a large sphere.

If we increase the area of the triangle to 500 units²:

$$\sum(Interior\ angles) = \pi + \frac{500}{1000^2} = \pi + \frac{500}{1000000} = \pi + 0.0005$$

Similarly, if we instead decrease the radius of the sphere to 3 units:

$$\sum(Interior\ angles) = \pi + \frac{0.1}{3^2} = \pi + \frac{0.1}{9} = \pi + 0.0125.$$

Thus, as the area of a triangle approaches the area of a hemisphere, we obtain

$$\sum(Interior\ angles) = \pi + \frac{2\pi r^2}{r^2} = \pi + 2\pi = 3\pi.$$
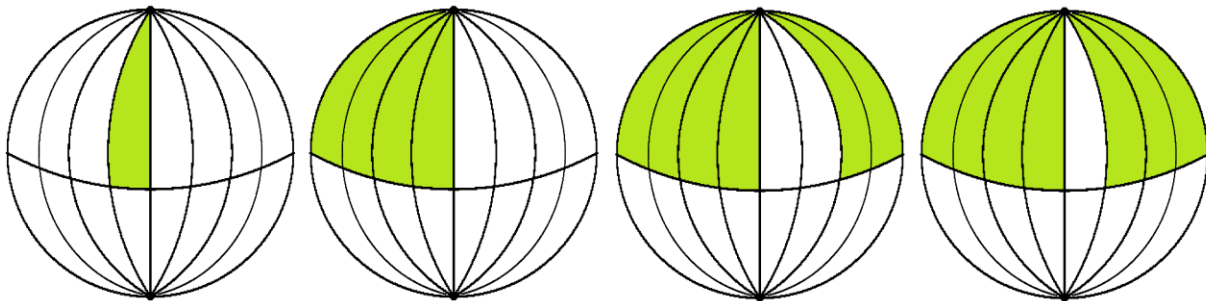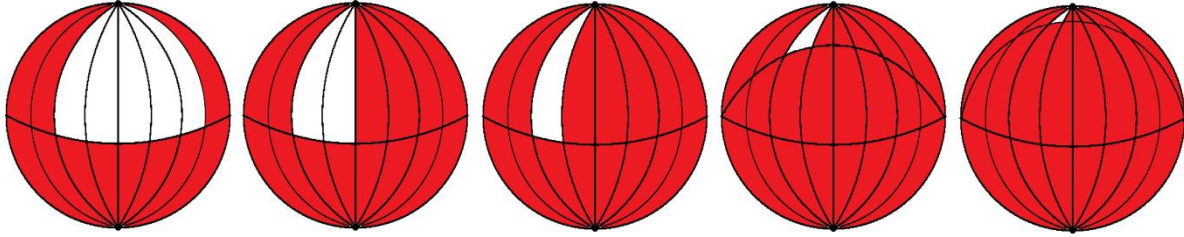

Figure 19 Triangle area approaching that of the hemisphere.

Finally, and somewhat trivially (taking the major triangle rather than the minor one)

$$\lim_{A(\triangle)\to 4\pi r^2} \sum("interior"\ angles) = 4\pi$$

25

Triangle area approaching the that of the sphere.

So, we have the set of equations:

$$\lim_{r \to \infty} \sum (interior\ angles) = \pi$$

$$\lim_{A(\triangle) \to 2\pi r^2} \sum (interior\ angles) = 2\pi$$

$$\lim_{A(\triangle) \to 4\pi r^2} \sum ("interior"\ angles) = 4\pi\,.$$

Note that in the first equation, the area of the triangle remains fixed, and in the latter two equations the radius is fixed.

VI.   Quadrilaterals
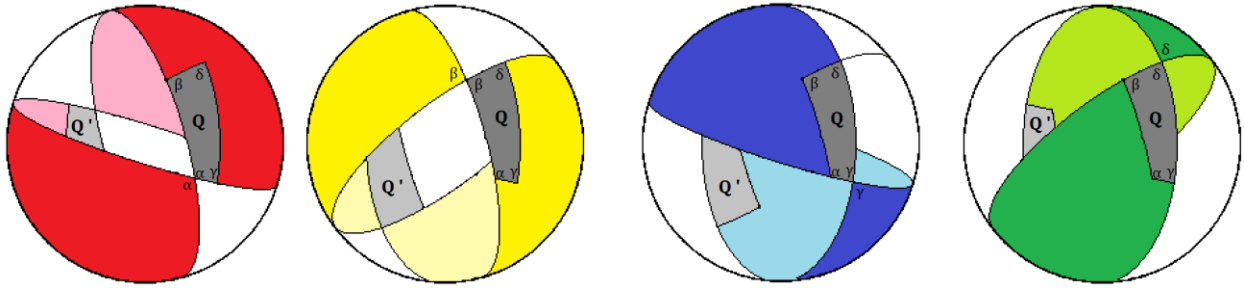### 1.   Girard's Theorem for Quadrilaterals

Girard's Theorem for triangles on an elliptic surface can also be extended to spherical quadrilaterals.

Let Q be a quadrilateral on a sphere of radius r having angles α, β, γ, and δ.

In the case of the quadrilateral, the extension of the theorem states:

$$\alpha + \beta + \gamma + \delta - 2\pi = \frac{A(Q)}{r^2},$$

where A(Q) is the area of the quadrilateral Q.

Figure 21 **The eight lunes associated with a spherical quadrilateral.**

Angle α defines a pair of congruent lunes (one lune containing Q and one containing Q'). Let the lune containing Q be denoted L_α and the one containing Q' be L'_α. Define the lunes constructed from the other angles similarly. See Figure 21.

The area of each of the congruent lunes is given by the following set of equations

$$A(L_\alpha) = A(L'_\alpha) = 2(\alpha)r^2$$
$$A(L_\beta) = A(L'_\beta) = 2(\beta)r^2$$
$$A(L_\gamma) = A(L'_\gamma) = 2(\gamma)r^2$$
$$A(L_\delta) = A(L'_\delta) = 2(\delta)r^2.$$

Since each of the eight lunes contains either Q or Q', the sum of the areas of all eight lunes accounts for twice the area of the sphere plus twice the area of Q and twice the area of Q', and is given by

27

$A(L_\alpha) + A(L'_\alpha) + A(L_\beta) + A(L'_\beta) + A(L_\gamma) + A(L'_\gamma) + A(L_\delta) + A(L'_\delta) = 2(4\pi r^2) + 2A(Q) + 2A(Q')$.

Since $A(Q) = A(Q')$

$$A(L_\alpha) + A(L'_\alpha) + A(L_\beta) + A(L'_\beta) + A(L_\gamma) + A(L'_\gamma) = 8\pi r^2 + 4A(Q)$$

And since $A(L_\alpha) = A(L'_\alpha)$, $A(L_\beta) = A(L'_\beta)$, $A(L_\gamma) = A(L'_\gamma)$, and $A(L_\delta) = A(L'_\delta)$,

$$2A(L_\alpha) + 2A(L_\beta) + 2A(L_\gamma) + 2A(L_\delta) = 8\pi r^2 + 4A(Q).$$

As the area of a lune of angle θ is given by $2(\theta)r^2$,

$$2(2(\alpha)r^2) + 2(2(\beta)r^2) + 2(2(\gamma)r^2) + 2(2(\delta)r^2) = 8\pi r^2 + 4A(Q).$$

And we have

$$4(\alpha)r^2 + 4(\beta)r^2 + 4(\gamma)r^2 + 4(\delta)r^2 = 8\pi r^2 + 4A(Q)$$

$$(\alpha)r^2 + (\beta)r^2 + (\gamma)r^2 + (\delta)r^2 = 2\pi r^2 + A(Q)$$

$$(\alpha)r^2 + (\beta)r^2 + (\gamma)r^2 + (\delta)r^2 - 2\pi r^2 = A(Q)$$

$$(\alpha) + (\beta) + (\gamma) + (\delta) - 2\pi = \frac{A(Q)}{r^2}.$$

Without loss of generality, we again assume we are on a unit sphere $(r = 1)$, so

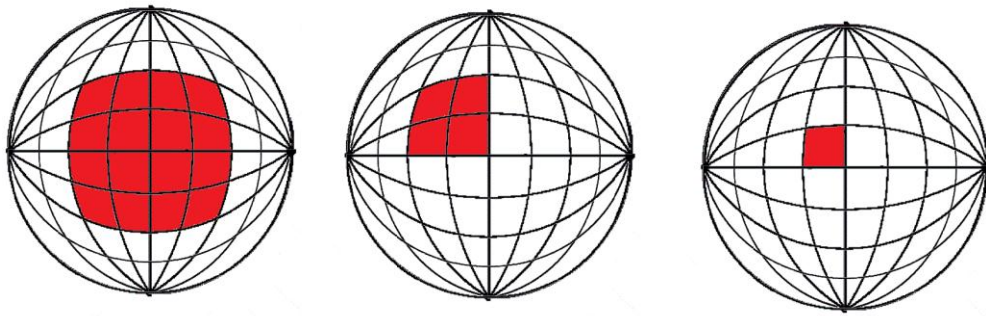$$(\alpha) + (\beta) + (\gamma) - 2\pi = A(Q),$$

as desired.

∎

2. Relative Size of Quadrilaterals on the Sphere

The behavior of quadrilaterals on the sphere is similar to that of triangles. That is, as the radius of the sphere increases and the area of the quadrilateral decreases, the quadrilateral behaves more similarly to its planar, Euclidean counterpart.

As we have just shown,

$$(\alpha) + (\beta) + (\gamma) + (\delta) - 2\pi = \frac{A(Q)}{r^2}.$$

As the radius of the sphere increases and the area of the quadrilateral decreases, we come to the following conclusion:

$$\lim_{\substack{A(Q)\to 0 \\ r\to\infty}} (\alpha) + (\beta) + (\gamma) + (\delta) = 2\pi.$$

Consider the sphere of radius 100000 units and the quadrilateral of area $A(Q) = 2$ units^2

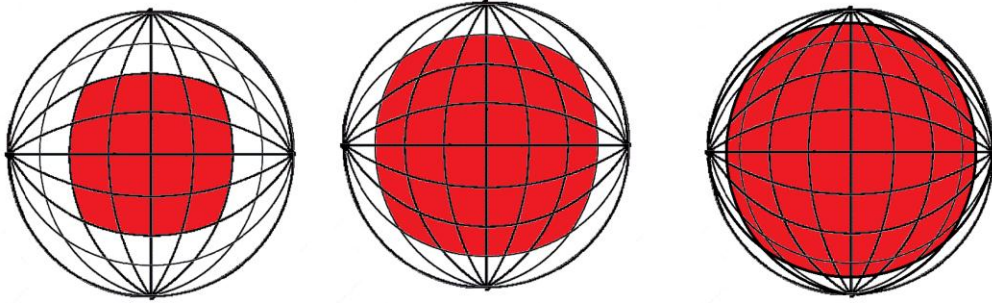$$(\alpha) + (\beta) + (\gamma) + (\delta) - 2\pi = \frac{2}{10^{12}}$$

Then

$$(\alpha) + (\beta) + (\gamma) + (\delta) = 2\pi + \frac{2}{10^{12}} = 2\pi + 2 \times 10^{-12} \approx 2\pi + 0.000000000002 \approx 2\pi.$$

Hence the angle sum in this case is not much larger than 2π, showing that small quadrilaterals on large spheres approach Euclidean limits. We thus have the following limit.

$$\lim_{r\to\infty} \sum (\alpha, \beta, \gamma, \delta) = 2\pi.$$

When quadrilaterals increase in area relative to the fixed surface area of the sphere on which they lie, they approach the area of a hemisphere.

Figure 23 Area of a quadrilateral approaching the area of the hemisphere.

$$\lim_{A(Q)\to 2\pi r^2} \sum(\alpha, \beta, \gamma, \delta) = 4\pi.$$

Taking the interior of the quadrilateral to be the larger area bounded by its sides, or extending the "growth" of the quadrilateral past the hemisphere mark, shows that the area of the quadrilateral may approach the area of the entire sphere.
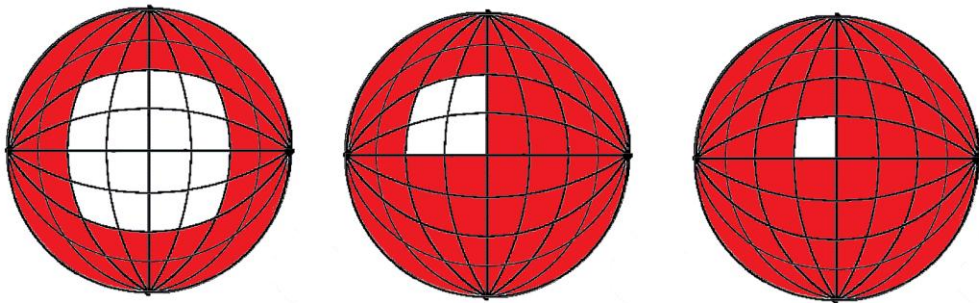
Figure 24 Area of a quadrilateral approaching the area of the sphere.

Hence

$$\lim_{A(Q)\to 4\pi r^2} \sum(\alpha, \beta, \gamma, \delta) = 6\pi.$$

3. Saccheri and Lambert Quadrilaterals

In elliptic geometry, a perpendicular may be constructed from any great circle, say *c,* by any arc of a great circle intersecting *c* and incident to the poles of *c*. See Figure 25.
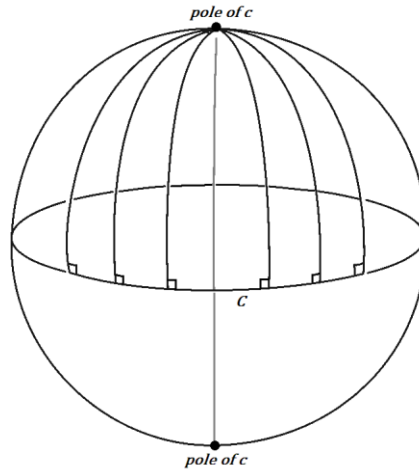

Figure 25 **All lines incidend with the poles of *c* are perpendicular to *c*.**

We call a quadrilateral ■ABCD in which the adjacent angles A and B are right angles *bi-right*. Label such quadrilaterals such that the first two letters represent the two adjacent right angles. The *base* of the quadrilateral will be defined as the side AB (the side joining the two right angles). The *summit* CD will be defined as the side which shares none of its vertices with the base. Angles C and D will be called the *summit angles* and arcs CA and DB will be called the *sides* of the bi-right quadrilateral ■ABCD.

A. Saccheri Quadrilaterals

An *isosceles bi-right quadrilateral* is one in which the sides are congruent (CA≅DB). Such a quadrilateral is called a *Saccheri quadrilateral.* See Figure 26. Originally, Saccheri defined these quadrilaterals in his unsuccessful effort to prove that Euclid's fifth postulate follows from the preceding four postulates. We now discuss several results about Saccheri quadrilaterals.
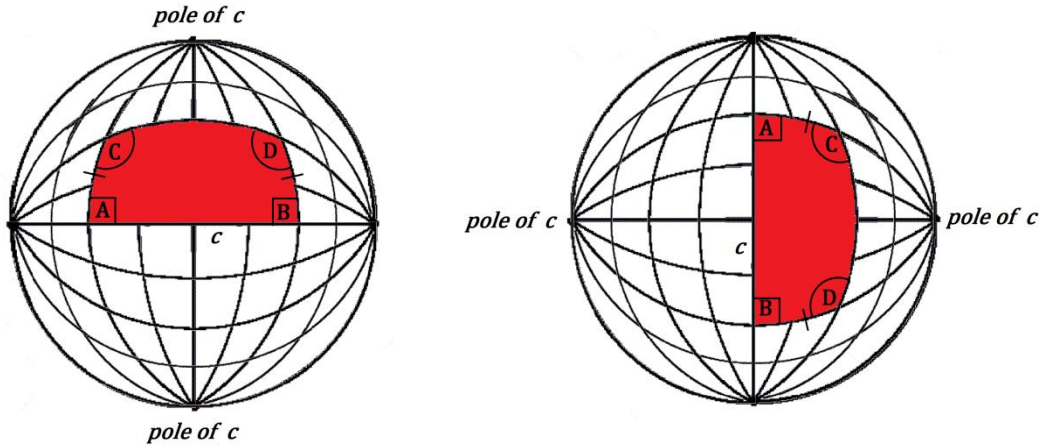
Figure 26 Saccheri quadrilateral on a sphere.

Theorem (Saccheri I): The summit angles of a Saccheri quadrilateral are congruent to one another. That is, in this construction, $\angle C \cong \angle D$.

Proof: By definition of a Saccheri quadrilateral, $CA \cong DB$. By SAS $\triangle CAB \cong \triangle DBA$. Then by SSS $\triangle DCB \cong \triangle CDA$. Hence, $\angle C \cong \angle D$. ∎
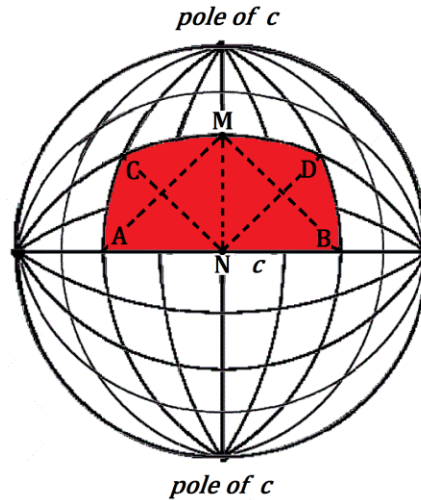

Figure 27 Proof of Saccheri II.

Theorem (Saccheri II): The line joining the midpoint of the base to the midpoint of the summit is perpendicular to both the base, and the summit.

Proof: See Figure 27 with Saccheri quadrilateral ■ABCD. Let N be the midpoint of the base and M be the midpoint of the summit. So, $CM \cong DM$ and $AN \cong BN$. Then $\triangle ACM \cong \triangle BDM$ by Saccheri I and SAS. Then $AM \cong BM$, as corresponding parts of congruent triangles are congruent. Thus, by SSS, $\triangle ANM \cong \triangle BNM$. But since $\angle ANM$ and $\angle BNM$ are supplementary and congruent, they must be

right angles. Similarly, we have $\Delta ACN \cong \Delta BDN$ by SAS, and it follows $\Delta CNM \cong \Delta DNM$ by SSS. Thus $\angle CMN \cong \angle DMN$. Since these angles are also supplementary, they are both right angles as well.

∎

Theorem (Saccheri III): In any bi-right quadrilateral, the greater side is opposite the greater angle. Hence the summit of a Saccheri quadrilateral on the sphere will always be shorter than the base.

Proof: Since the Saccheri quadrilateral has two right angles, we have the following equation from Girard's theorem for quadrilaterals:

$$\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) + (\angle C) + (\angle D) - 2\pi = \frac{A(Q)}{r^2}.$$

Let the radius be 1 and let the area of the quadrilateral be (necessarily) greater than zero.

Then we have

$$(\angle C) + (\angle D) - \pi = A(Q) > 0$$

$$(\angle C) + (\angle D) - \pi > 0$$

Since A(Q) must be positive, the sum of the remaining two angles, $(\angle C) + (\angle D)$, must be greater than π. As we have proven that the two summit angles are congruent $(\angle C) = (\angle D)$, then without loss of generality, we have the following set of inequalities

$$2(\angle C) - \pi > 0$$

$$2(\angle C) > \pi$$

$$(\angle C) > \frac{\pi}{2}.$$

The summit angles of the Saccheri quadrilateral on the sphere are strictly greater than $\frac{\pi}{2}$, as desired.

∎

B. Lambert Quadrilaterals

We define a *Lambert quadrilateral* as a tri-right quadrilateral. A Lambert quadrilateral is dissimilar to a Saccheri quadrilateral in that the summit angles may not be congruent in any given geometric model. Indeed, in the case of elliptic geometry, they are necessarily *not* congruent.
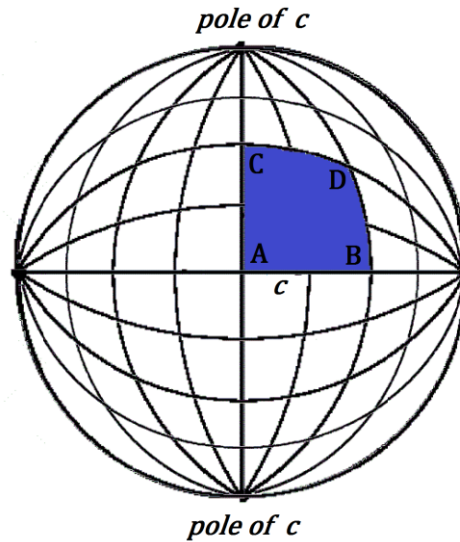

<div align="center">**Figure 28** **A Lambert quadrilateral.**</div>

To prove that Lambert quadrilaterals exist on the elliptic sphere, we construct one.

Let *c* be any great circle on the sphere with poles C and C' (any great circle incident with these polls will necessarily be perpendicular to *c*). In Figures 29-31 we simplify Figure 28. In all such figures only the front-facing hemisphere of the sphere is shown.
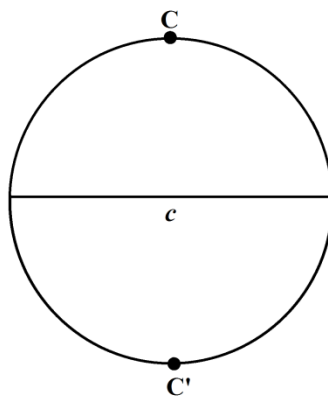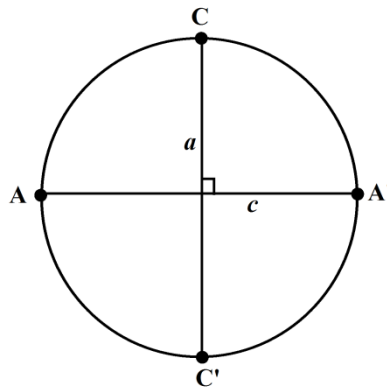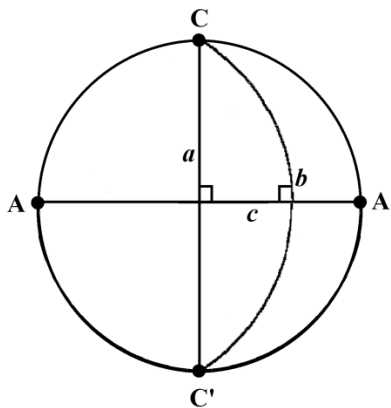

<div align="center">**Figure 29** **The line *c* and its poles C and C'.**</div>

We construct a great circle, *a*, such that *a* passes through C and C'. We label the poles of *a* as A, and A' as in Figure 30.

Figure 30 The line *a* perpendicular to *c*, with poles A and A'.

We then construct another great circle, *b,* such that *b* is also incident with C and C' and $b \neq a$. See Figure 31.

Figure 31 Construction of *b*.

Finally we construct a great circle, *d,* such that *d* is incident with A and A' and $d \neq c$. Recall that any arbitrary great circle incident with A and A' will be perpendicular to *a*. See Figure 32.
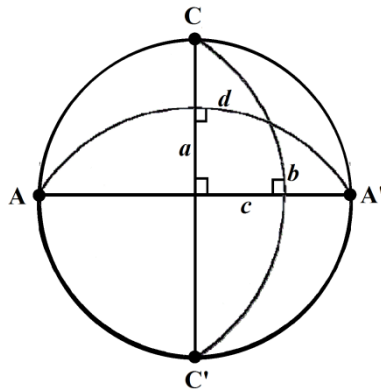
Figure 32 Construction of *d* and the resulting Lambert quadrilateral.

The inscribed quadrilateral is a tri-right quadrilateral. Hence Lambert quadrilaterals exist in elliptic geometry.

Thereom: The fourth angle of a Lambert quadrilateral will have measure greater than $\frac{\pi}{2}$.

Proof: Again we refer to Girard's theorem for quadrilaterals and use Figure 33 to obtain

$$(\sphericalangle A) + (\sphericalangle B) + (\sphericalangle C) + (\sphericalangle D) - 2\pi = \frac{A(Q)}{r^2}.$$
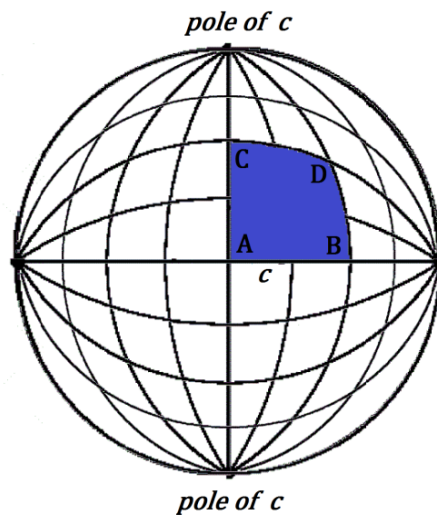
Figure 33 Lambert quadrilateral.

For simplicity, we again use the unit sphere.

$$(\sphericalangle A) + (\sphericalangle B) + (\sphericalangle C) + (\sphericalangle D) - 2\pi = A(Q)$$

Since the quadrilateral is tri-right, and must have positive area:

$$\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) + (\sphericalangle D) - 2\pi > 0$$

$$\left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) + \left(\frac{\pi}{2}\right) + (\sphericalangle D) > 2\pi$$

$$\left(\frac{3\pi}{2}\right) + (\sphericalangle D) > 2\pi$$

$$(\sphericalangle D) > \left(\frac{\pi}{2}\right).$$

The remaining angle is obtuse, as desired.

$$\blacksquare$$

A direct result from our observations of both Saccheri and Lambert quadrilaterals is that rectangles do not exist in spherical geometry. If this result seems fantastic or counterintuitive at first, one may simply recall the elliptic parallel postulate; that there are no parallel lines in elliptic geometry.

Chapter 3: Conclusion

I.   Summary

We live on a sphere rather than on a Euclidean plane, and thus it is important for us to understand the geometry of the earth we live on. Unfortunately, although Euclidean geometry is convenient (since the earth is so large and we are so small, our everyday experience is closer to Euclidean geometry), if airline pilots restricted their view of the world to a Euclidean model, we would never arrive at our intended destination. We must better understand the rules that govern the geometry of the earth, and such an understanding involves elliptic geometry. By altering the axioms and assumptions from Euclidean geometry, we can derive a geometry where taking a pair of antipodal points as a "point" and great circles as "lines" on the surface a sphere is a consistent model.

Beganing by discussing the axioms from neutral geometry and then adapting them to the spherical model of elliptic geometry, we proceeded to examine polygons on the sphere and Girard's Theorem, relating spherical polygons to their Euclidean counterparts. We discussed consequences of Girard's theorem, including the distortion of maps, and observed that small geometric figures on large spheres have behavior approaching Euclidean parameters.

II.   Suggestions for Further Study

Though the unexplored specifics of elliptic geometry are vast, nuanced and numerous, here are three primary topics we marked for future exploration.

- In this paper, we discussed only polygons with four or fewer sides. It appears evident that Girard's Theorem would apply to arbitrary polygons of higher order. We hypothesize that the spherical excess relative to the expected Euclidean angle sum will become less pronounced as the number of sides of the polygon increases, however time did not allow for detailed analysis of higher order polygons. In future, it would be of interest to explore these polygons and to discuss the existence and parameters of regular polygons in elliptic geometry. We are able to construct equilateral and equiangular triangles and quadrilaterals in elliptic geometry, however these polygons show considerable distortion when compared to their Euclidean counterparts.

- A discussion of the existence and constraints of circles in elliptic geometry would also be valuable. By definition, a circle is the collection of points on the sphere equidistant from any given point. The latitude lines on a globe are examples of circles with center at the North or South Pole. It would be of interest to discuss the area of a circle on the sphere relative to the area of its Euclidean counterpart with the same radius. Perhaps Girard's Theorem can be extended and generalized to circles as well.

- The focus of this paper has been exclusive to the spherical model of elliptic geometry. There are additional models for elliptic geometry which would likely prove informative and fascinating. For example, the Klein conformal elliptic model where angles are accurately represented as Euclidean angles [Greenberg 545].

- Both the spherical model and the Klein model are two-dimensional representations of elliptic geometry. Though a sphere is three-dimensional, this survey restricted study to the two-dimensional surface. It would be interesting to explore higher dimensional models.

III.   References

[1] Coxeter, H. S. M. 1998. *Non-Euclidean Geometry,* Washington, D.C.: The Mathematical Association of America (Incorporated)

[2] Greenberg, Marvin Jay. 2008. *Euclidean and Non-Euclidean Geometries,* New York: W.H. Freeman and Company.

[3] Honner, Patrick. "Sherical Geometry Explorer" Mr Honner - Math Appreciation. 17 March 2012, 12 August, 2012 <http://mrhonner.files.wordpress.com/2012/03/spherical-triangle.jpg>

[4] Little, David. "Spherical Geometry Explorer." 17 March 2012, 09 October 2012 <http://www.math.psu.edu/dlittle/java/geometry/spherical/toolbox.html>

[5] Norton, John D., "Spaces of Constant Curvature," Einstein for Everyone. 01 March 2010. 15 September 2012. <http://www.pitt.edu/~jdnorton/teaching/HPS_0410/chapters/non_Euclid_constant/5none_perp_12.gif>

[6] Polking, John C. "The Geometry of the sphere." 28 January 2000, 09 October 2012 <http://math.rice.edu/~pcmi/sphere/>

[7] Weisstein, Eric W. "Girard's Spherical Excess Formula," MathWorld- A Wolfram Web Resource. 30 October 2012, 09 October 2012 <http://mathworld.wolfram.com/GirardsSphericalExcessFormula.html>

[8] Weisstein, Eric W. "Spherical Excess." MathWorld--A Wolfram Web Resource. 30 October 2012, 09 October 2012 <http://mathworld.wolfram.com/SphericalExcess.html>

[9] Wilson, Edwin Bidwell. *Spherical Geometry*, American Mathematical Monthly **11** No. 1 (1904), 1-6.

[10] "File:Spherical triangle.svg," Wikipedia. 14 July 2008, 12 August 2012 <http://en.wikipedia.org/wiki/File:Spherical_triangle.svg>

[11] "Navigation and Chart Work – Chart Projections, " Sailtrain.co.uk. 01 November 2012 < http://www.sailtrain.co.uk/navigation/projections.htm>

[12] "Spherical Trigonometry," Wikipedia. 04 November, 2012, 12 August 2012 <http://en.wikipedia.org/wiki/Spherical_trigonometry>